Outline of the talk

Metric Embeddings into Trees

Arnold Filtser

Bar-Ilan University

May 06, 2024

Metric space

A metric space is an ordered pair (X, d_X), where X is a set and $d_X : X \times X \to \mathbb{R}_{\geq 0}$ is a function such that:

1. **Identity**: $\forall x, y \in X,\ d_X(x, y) = 0 \iff x = y$.
2. **Symmetry**: $\forall x, y \in X,\ d_X(x, y) = d_X(y, x)$.
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Examples:

- Weighted graph $G = (V, E, w)$ with shortest path distance.

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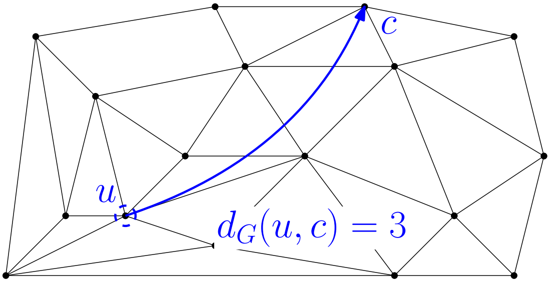

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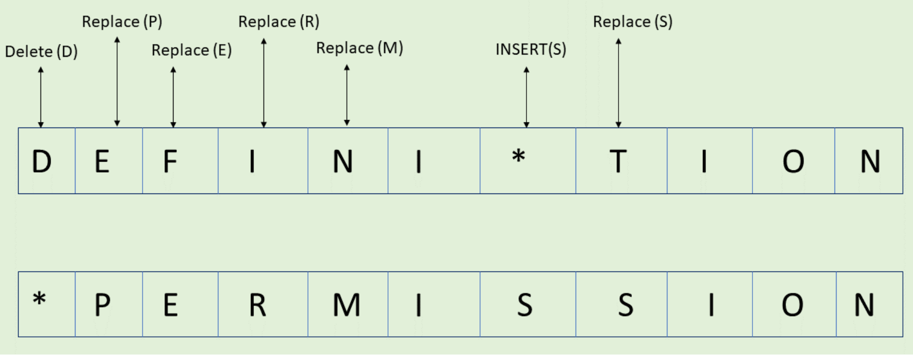

DEFINITION
EFINITION
PFINITION
PEINITION
PERNITION
PERMI–TION
PERMISTION
PERMISSION

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- Euclidean space ℓ_2 in $\mathbb{R}^d$: $d_{\ell_2}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2 = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$.

$$\left\| \begin{pmatrix} 5 \\ 8 \\ -3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\|_2 = \sqrt{\underbrace{|5-1|^2}_{16} + \underbrace{|8-10|^2}_{4} + \underbrace{|(-3)-1|^2}_{16} + \underbrace{|4-1|^2}_{9} + \underbrace{|1-3|^2}_{4}} = 7$$

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- Metric TSP.
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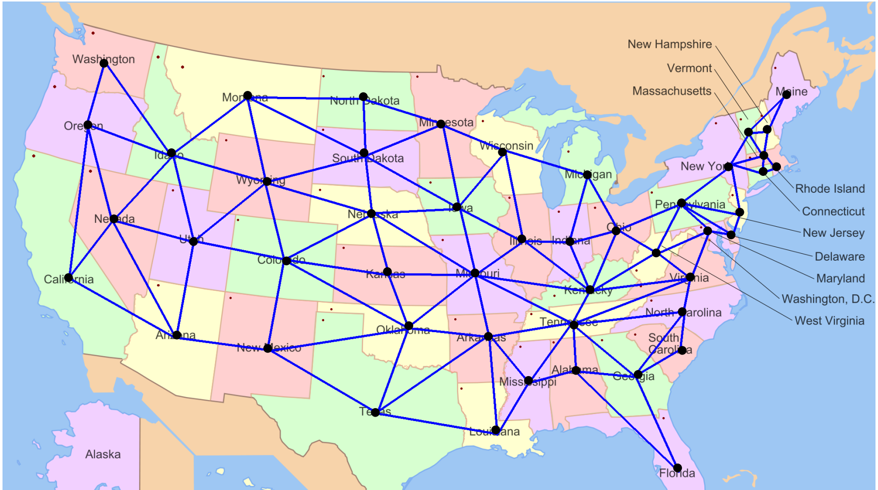
Given a metric space (X, d_X) find a permutation $x_0, x_1, \ldots, x_{n-1}$ of the points in X minimizing $\sum_{i=0}^{n-1} d_X(x_i, x_{i+1})$ (i.e. a Hamilton cycle of minimum weight).



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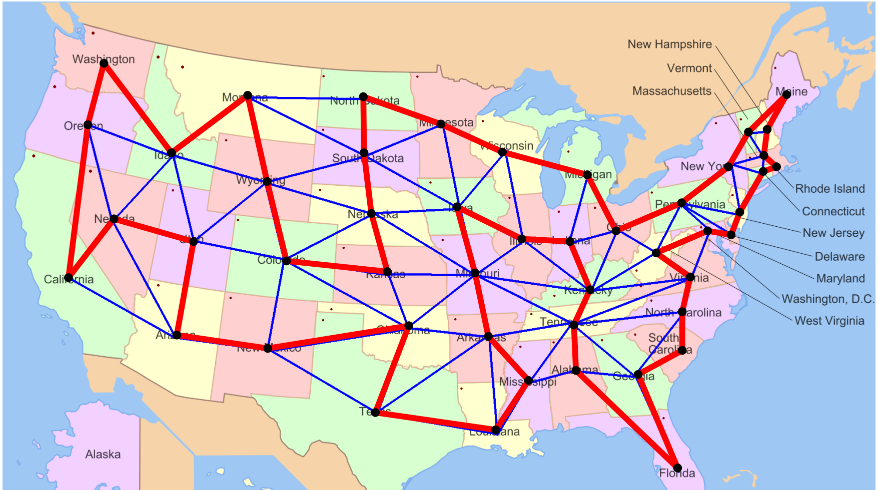

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NP-hard: a large class of equivalent problems (i.e. if you solved one-you solved all) for which we don't know of any efficient algorithms. It is generally believed that there are no efficient algorithms for these problems.

Theorem (Karp's list of 21 problems [Karp72])

The following problems are NP-Complete:

1. *SAT*
2. *0–1 integer programming*
3. *Clique*
4. *Set packing*
5. *Vertex cover*
6. *Set covering*
7. *Feedback node set*
8. *Feedback arc set*
9. *Directed Hamilton circuit*
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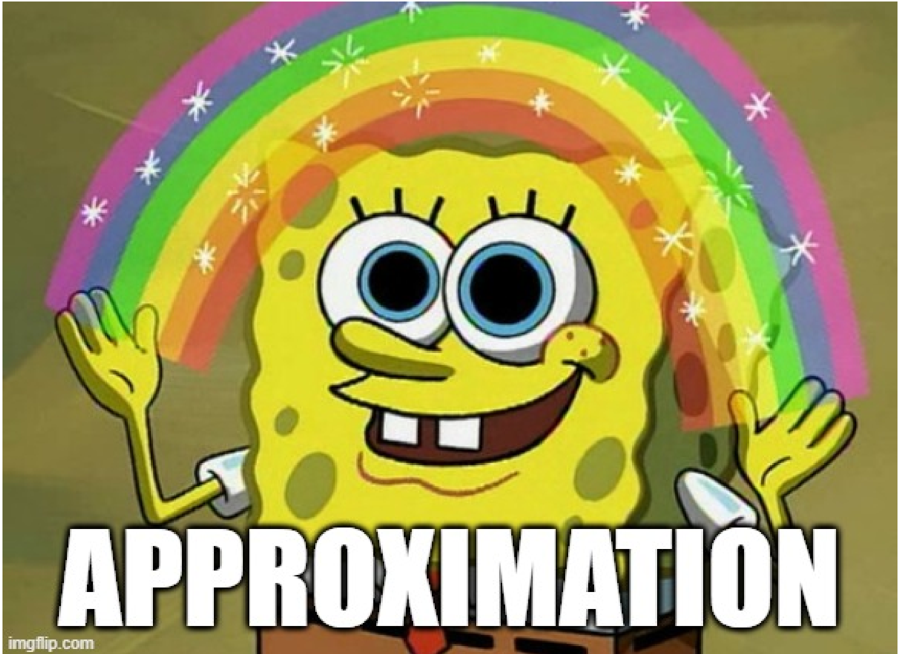
How should we cope with NP-hard problems?


APPROXIMATION
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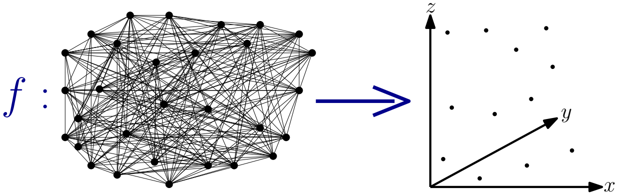
Metric Embeddings

Embedding

$(X, d_X), (Y, d_Y)$ metric spaces. $f : (X, d_X) \to (Y, d_Y)$ is called an **embedding**.



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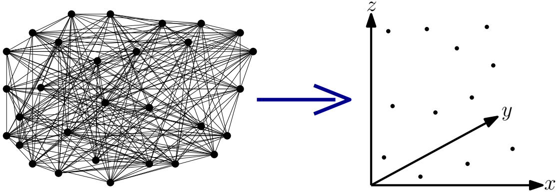

Preserve (approxierly) properties of the original space:

- **Distances**
- Cuts, Flows
- Commute time
- Effective resistance
- Clustering statistics.
- etc.

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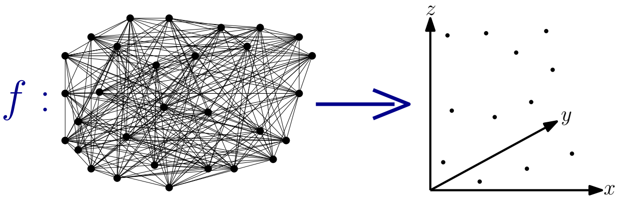

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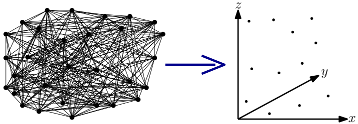
f : 

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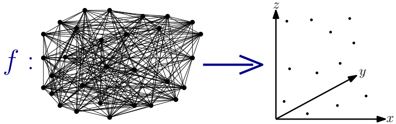

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So that we could run efficient algorithms on it...

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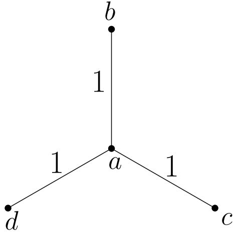

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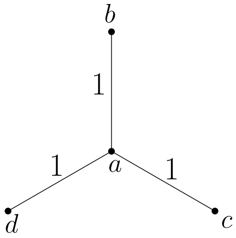

$\longrightarrow$

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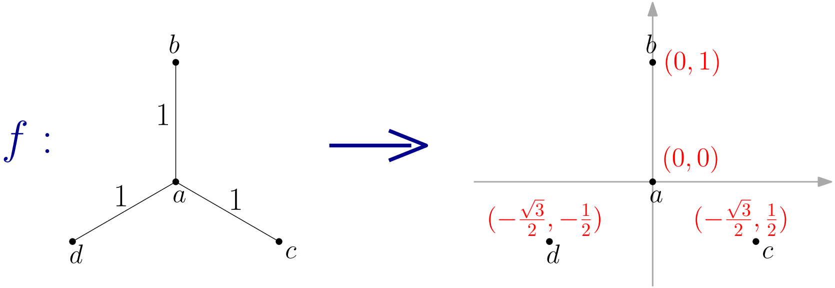

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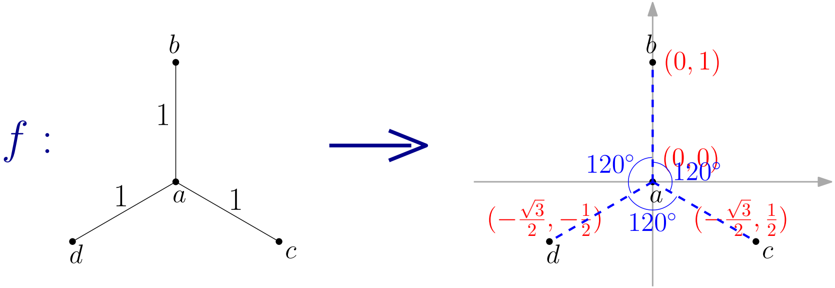

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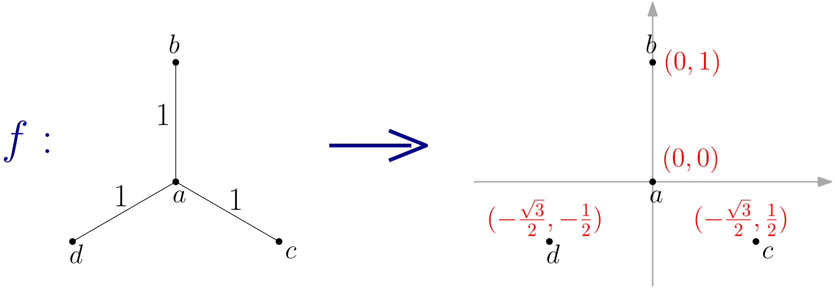

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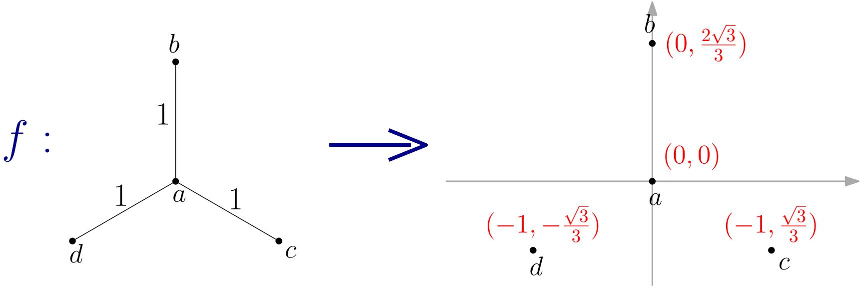

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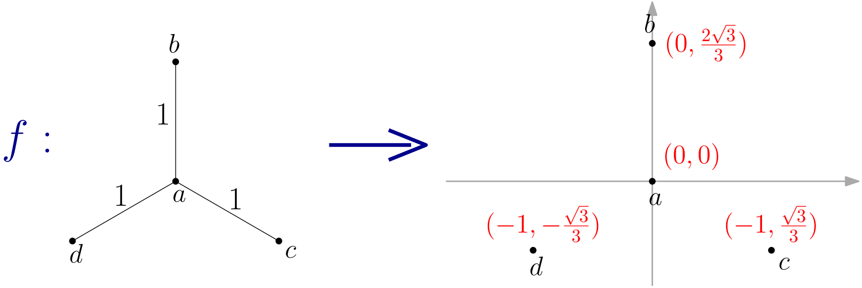

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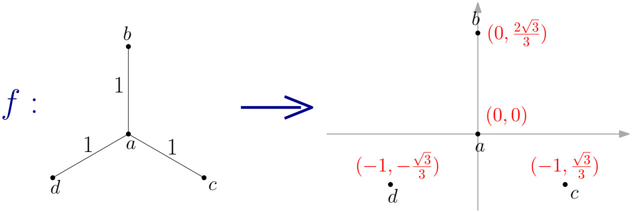

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The distortion of the embedding is $\frac{2}{\sqrt{3}} \approx 1.1547$.

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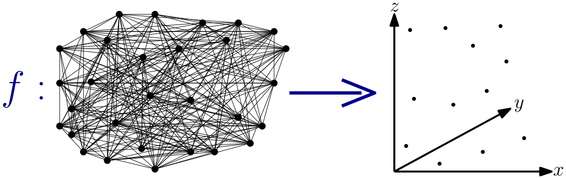

Theorem ([Bourgain 85])

*Every n-point metric (X, d_X) is **embeddable** into <u>Euclidean</u> space $(\mathbb{R}^d, \|\cdot\|_2)$ with **distortion** $O(\log n)$.*

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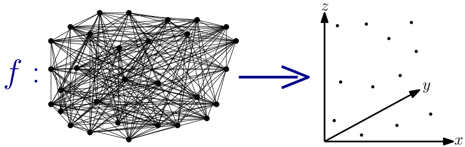

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[Bou85] is tight.

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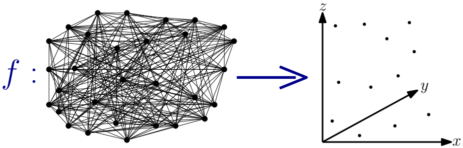
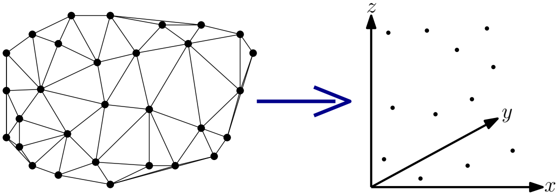


Applications:

- Approximation algorithms (e.g. **sparsest cut**, min graph bandwidth)
- Parallel computation (e.g. SSSP in MPC)
- Computational Biology (e.g. clustering and detecting protein seq.)
- etc.

Theorem ([Rao 99])

*Every n-point **planar** metric (X, d_X) is **embeddable** into <u>Euclidean</u> space $(\mathbb{R}^d, \|\cdot\|_2)$ with **distortion** $O(\sqrt{\log n})$.*

$f:$ 

Planar metric- the shortest path metric of a planar graph.

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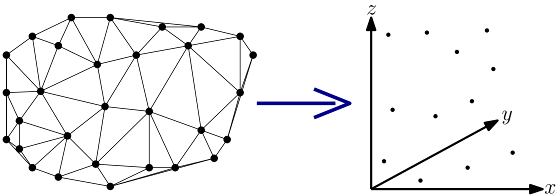
$f:$ 

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Theorem ([Newman, Rabinovich 03])

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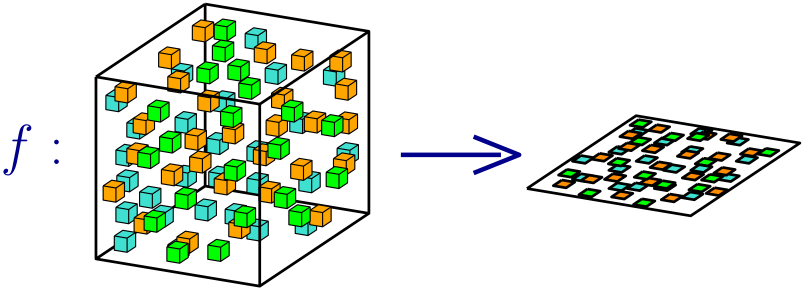
Embedding

$(X, d_X), (Y, d_Y)$ metric spaces. $f : (X, d_X) \to (Y, d_Y)$ is called an **embedding**.



Theorem ([Johnson, Lindenstrauss 84], Dimension Reduction)

$X \subset (\mathbb{R}^d, \|\cdot\|_2)$ set of size n. Then X embeds into $O(\log n/\epsilon^2)$ dimensional Euclidean space with distortion $1+\epsilon$.

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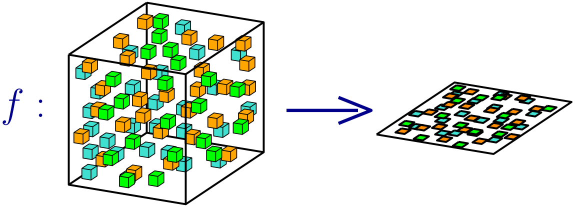

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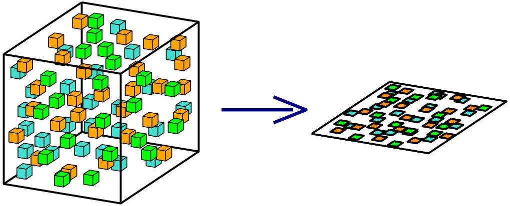
f : 

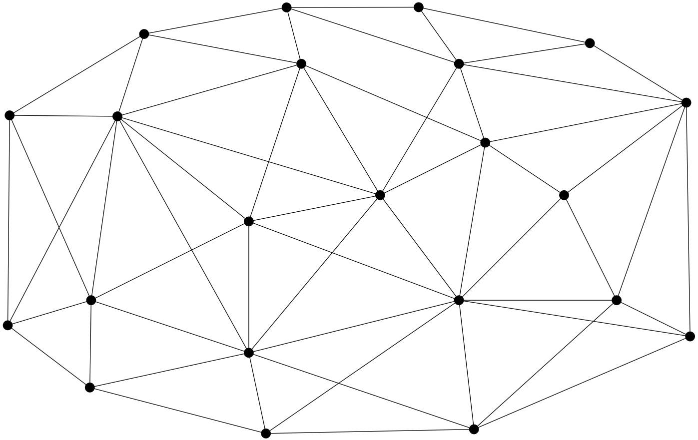
Applications:

- Speeding up-computation
- Clustering
- Nearest Neighbor Search
- Machine Learning
- etc.

Graph Spanners

$G = (V, E, w)$ weighted graph, a t-**spanner** is a **subgraph** $H = (V, E_H)$

s.t. $\forall u, v \in V, \quad d_H(u, v) \leq t \cdot d_G(u, v)$



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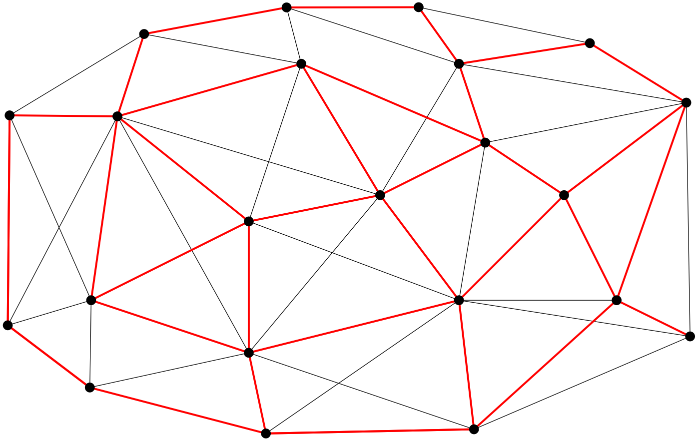

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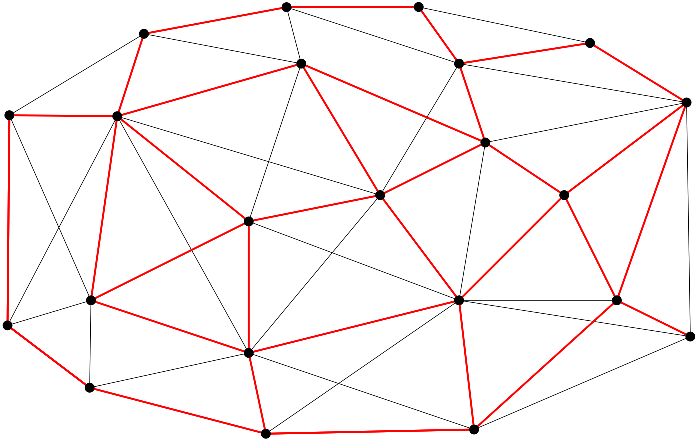

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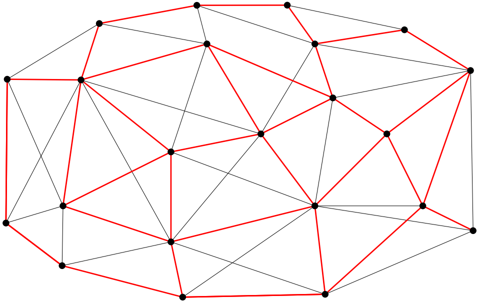

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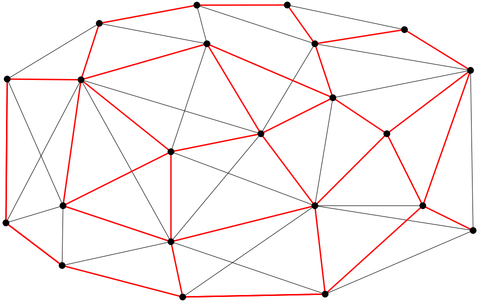

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Applications:

- Approximation Algorithms (e.g. PTAS for TSP)
- Distributed Computing
- Network Routing
- Computational Biology (e.g. measure genetic distance)
- etc.

Outline of the talk

Embedding into Trees

Tree is very **simple** and **desirable** target space.

Many NP-hard problems are easy on trees (using dynamic programming).

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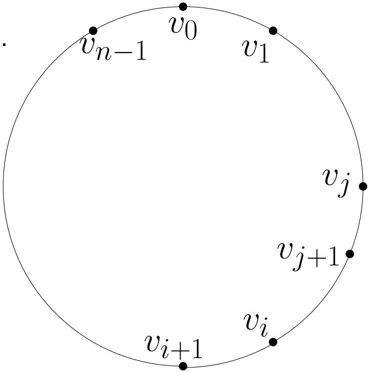

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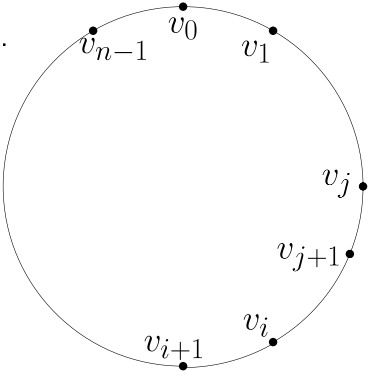

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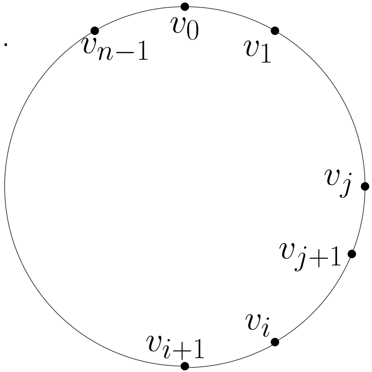

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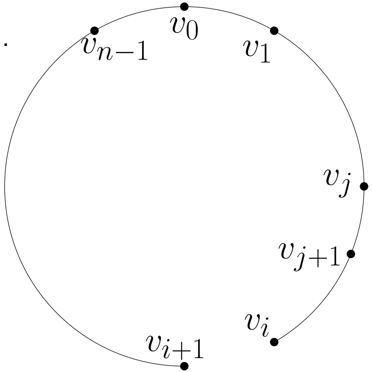

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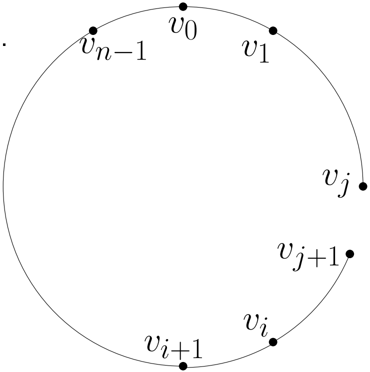

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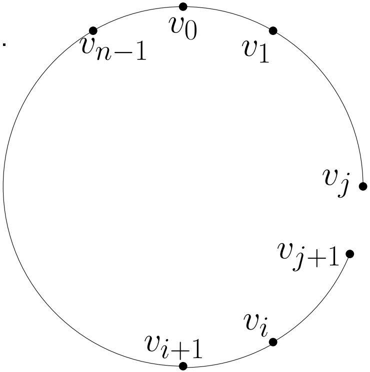

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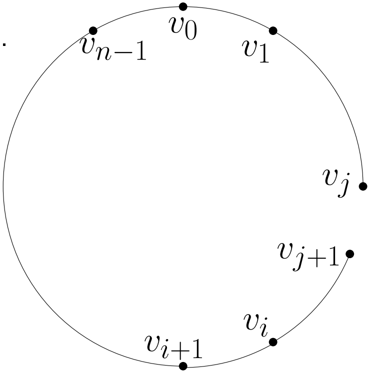

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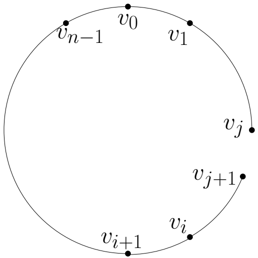

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By triangle inequality and linearity of expectation

$$\forall v_i, v_j, \quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(v_i, v_j)] = \sum_{q=i}^{j-1} \mathbb{E}_{T\sim\mathcal{D}}[d_T(v_q, v_{q+1 \pmod n})] \leq 2 \cdot d_{C_n}(v_i, v_j) \ .$$

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.

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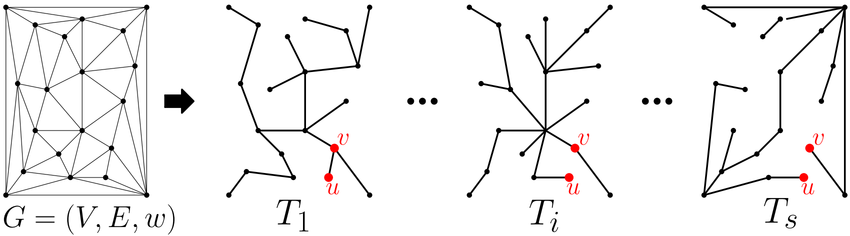

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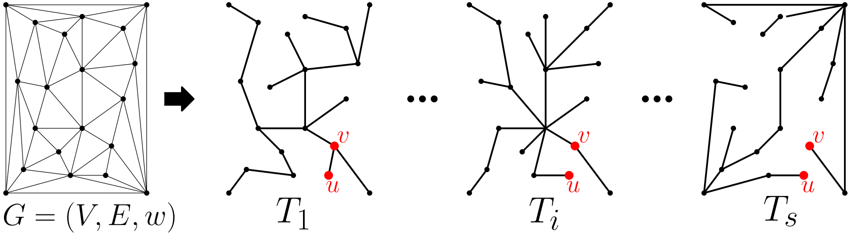

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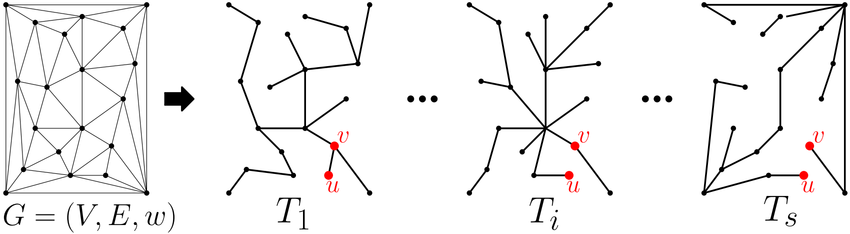

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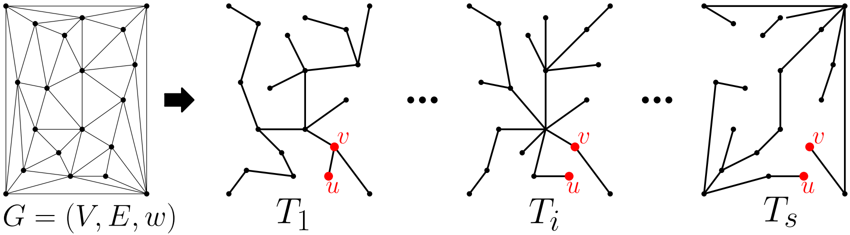

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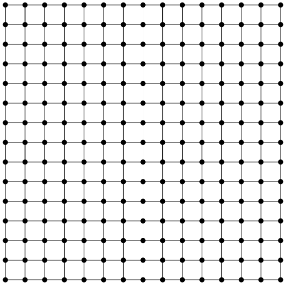
In fact, tight already for the $n \times n$ grid graph!



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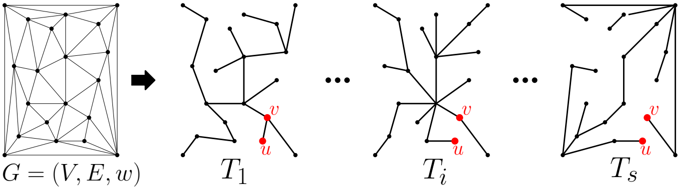

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Transforms arbitrary metric into a tree!

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Applications:

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Outline of the talk

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A **succinct** data structure that **approximately** answers distance queries.

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The properties of interest are size, distortion and query time.

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Sample $s = 4\log n$ trees $T_1, \dots, T_s$. Given x, y return $\mathrm{DO}(x,y) = \min_{i\in[1,s]} d_{T_i}(x,y)$.

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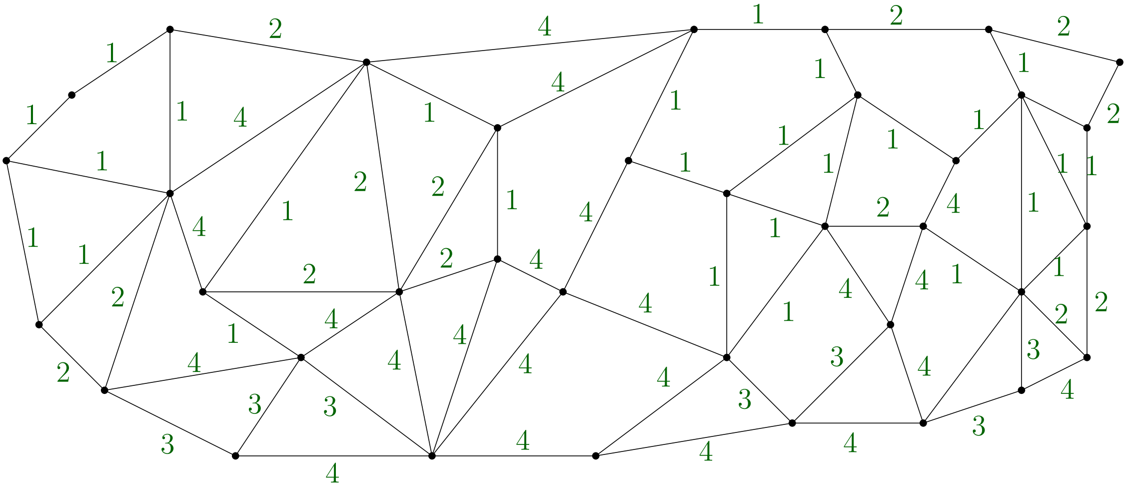
Distance oracle with approximation $2k - 1$, space $O(n^{1+\frac{1}{k}})$, and query time $O(1)$.

Outline of the talk

Minimum Spanning Tree (MST)

Given a weighted graph $G = (V, E, w)$ find a spanning tree if minimum total weight.



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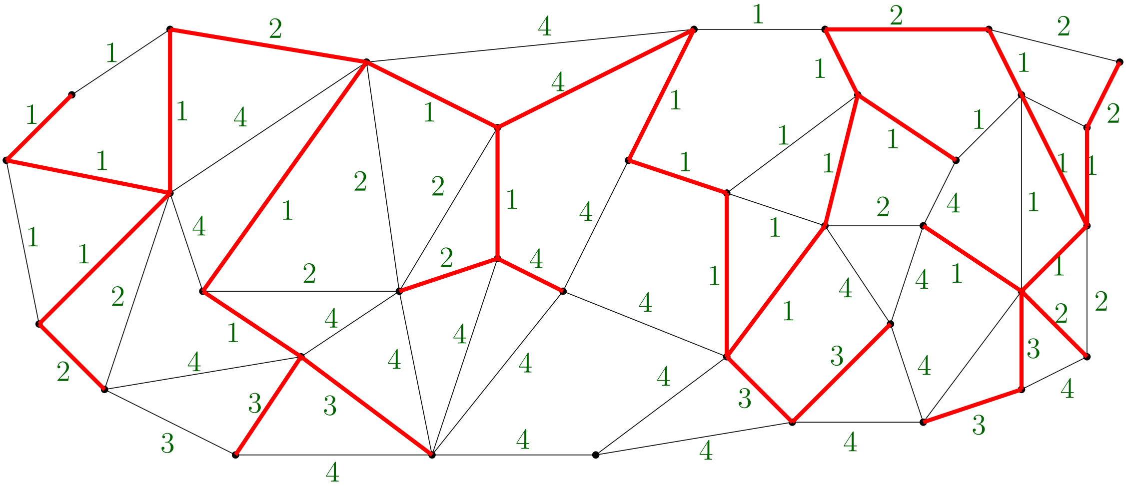

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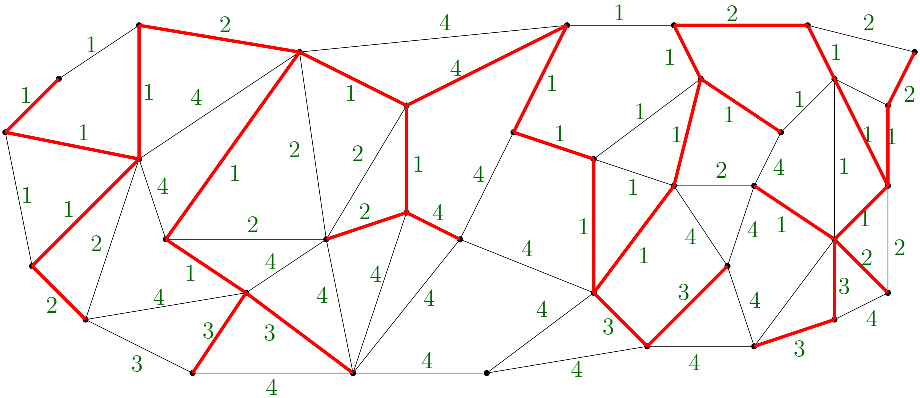
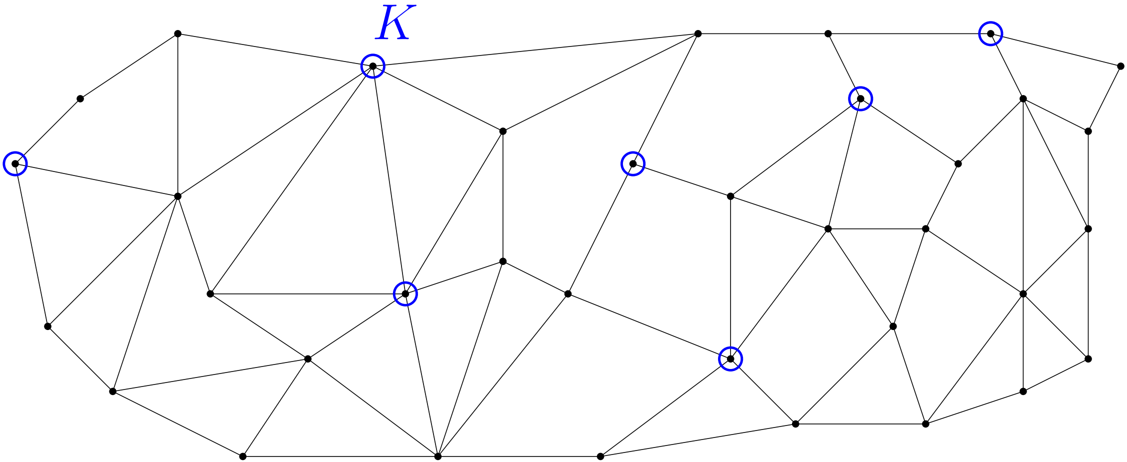


Classic problem, admits efficient poly-time solution.

Steiner Tree

Given set of terminals K, find minimum weight tree T spanning K



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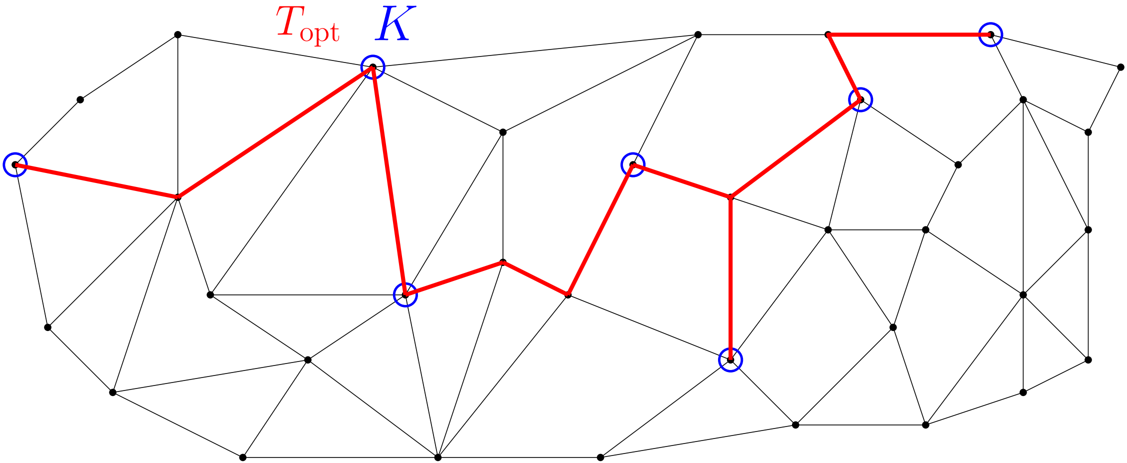

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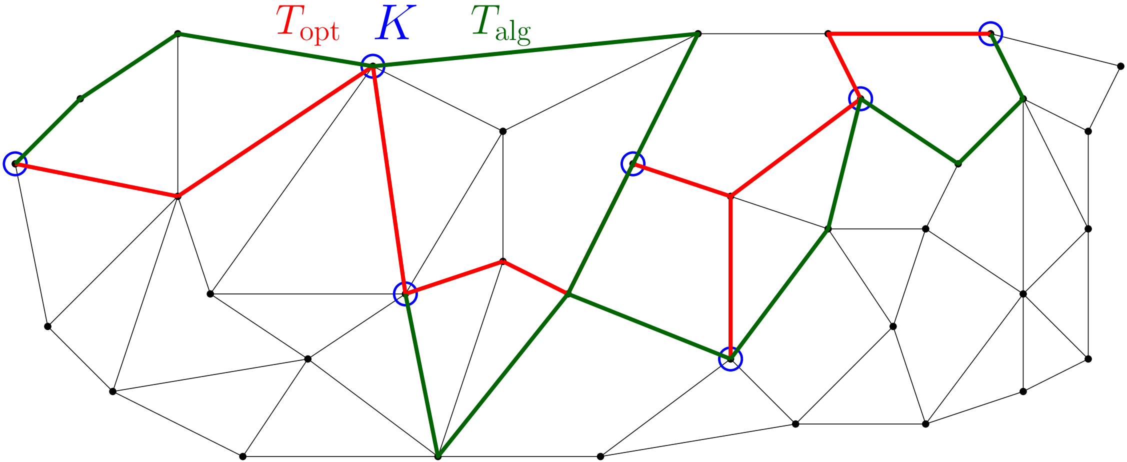

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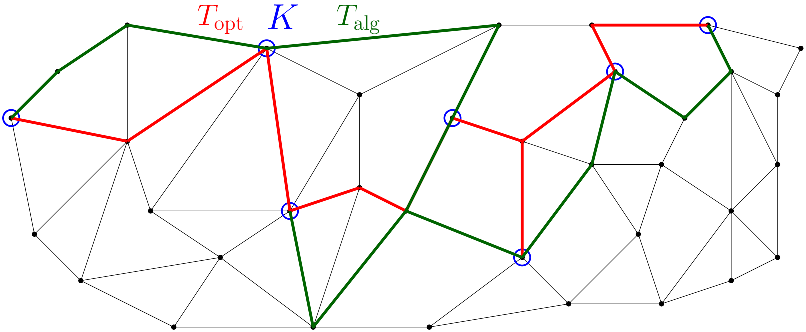
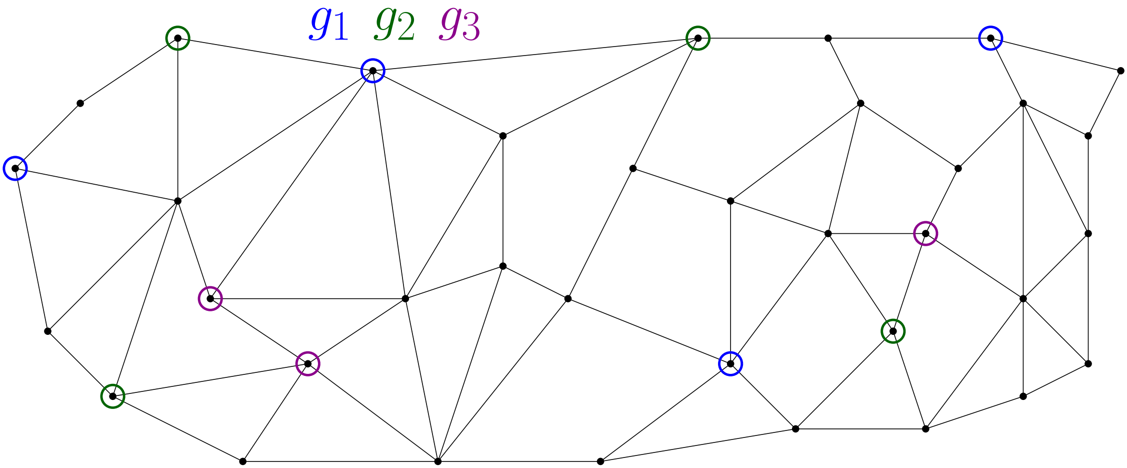


NP-hard. There is a simple 2-approximation algorithm for the Steiner tree problem.

That is, there is a polynomial time algorithm that returns a tree T_{alg} of weight at most $w(T_{\text{alg}}) \leq 2 \cdot w(T_{\text{opt}})$.

Group Steiner Tree (GST)

Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i



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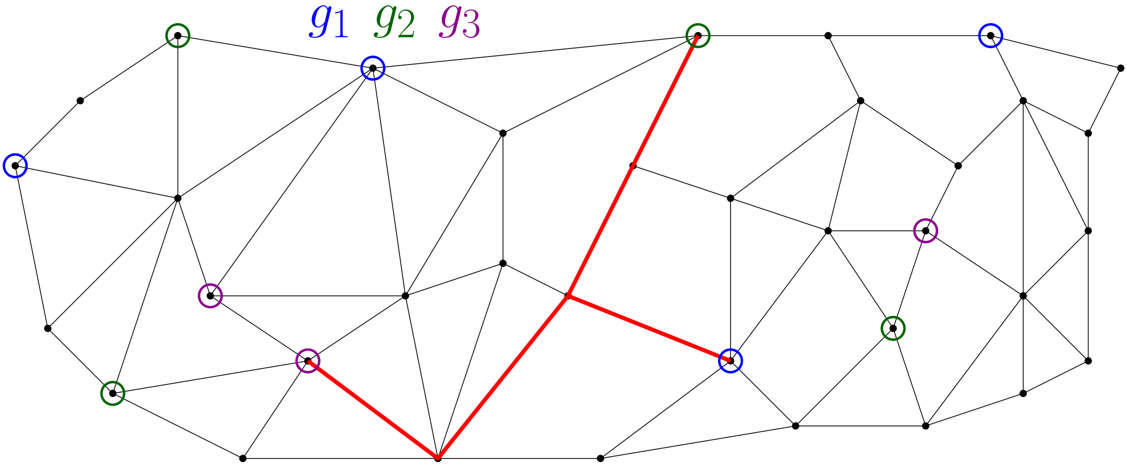

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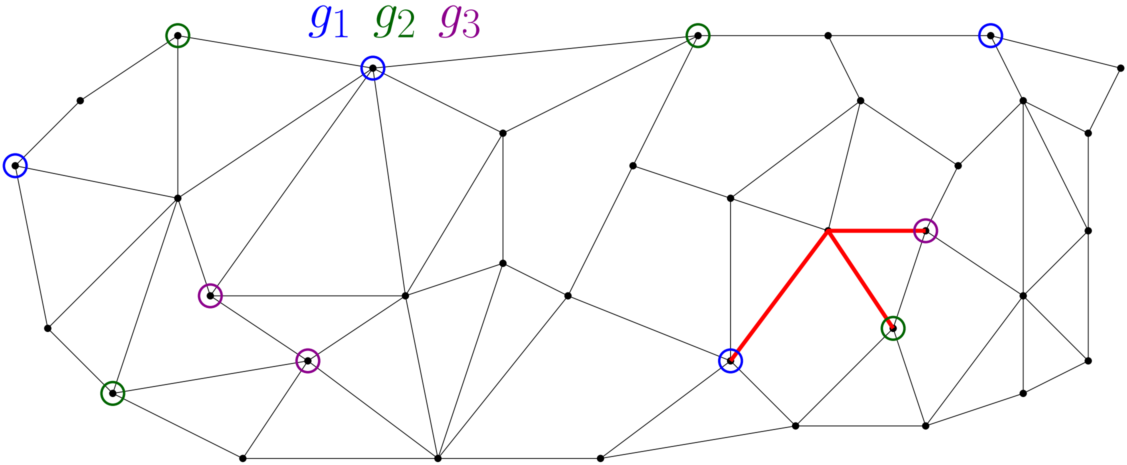

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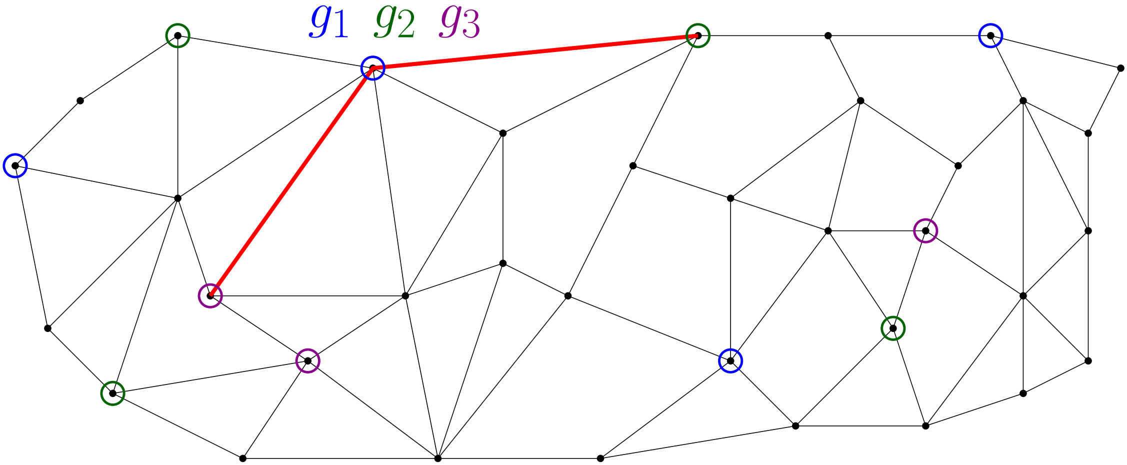

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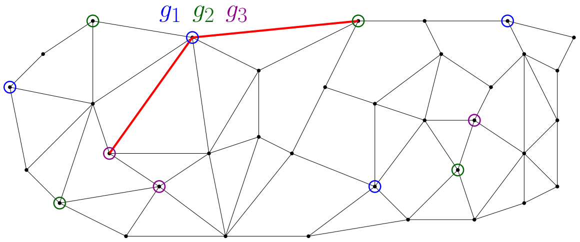

Note that Steiner tree is a special case of GST where all group sizes are 1.

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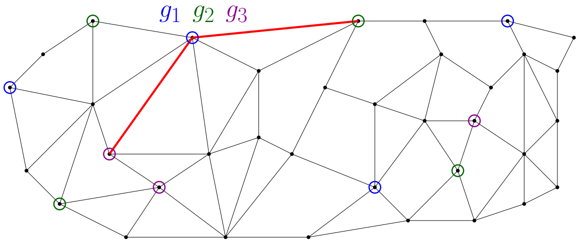

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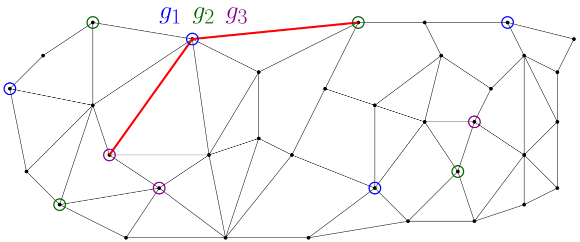

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We will use stochastic tree embeddings to generalize [GKR00] to general graphs.

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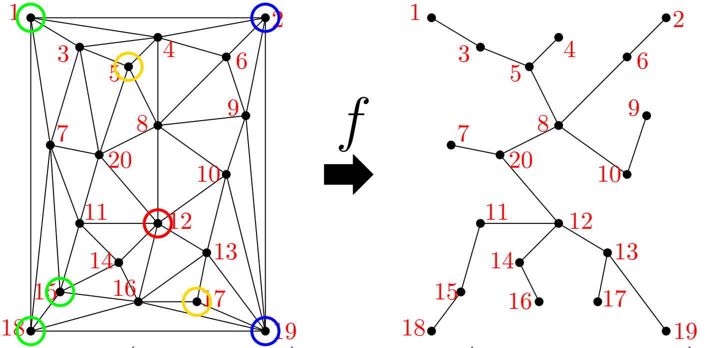
Embedding f with expected distortion $O(\log n)$.



$G = (V, E, w)$ $\quad$ $T = (X, E_T, w_T)$

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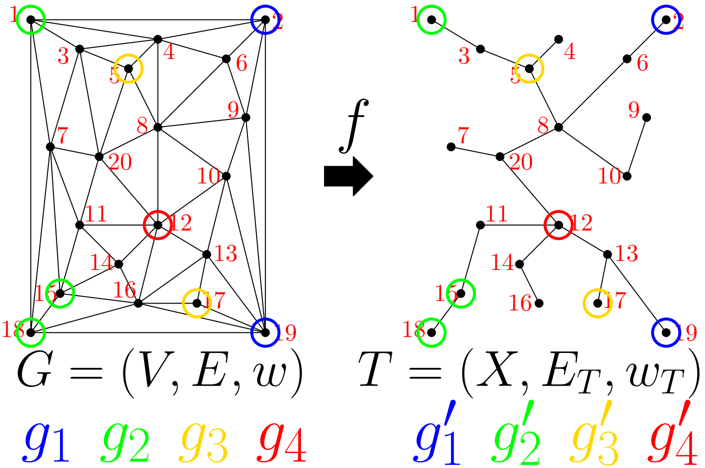

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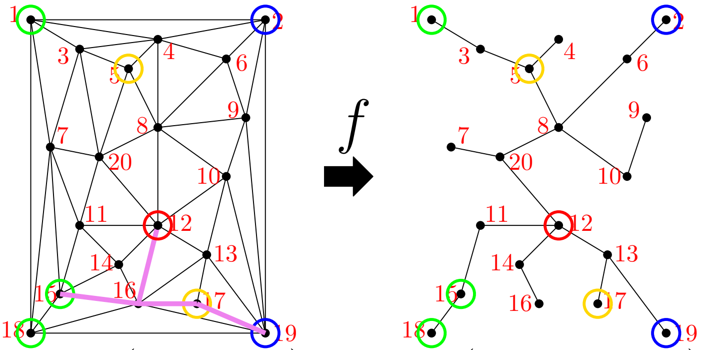

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$g_1 \quad g_2 \quad g_3 \quad g_4 \qquad g_1' \quad g_2' \quad g_3' \quad g_4'$

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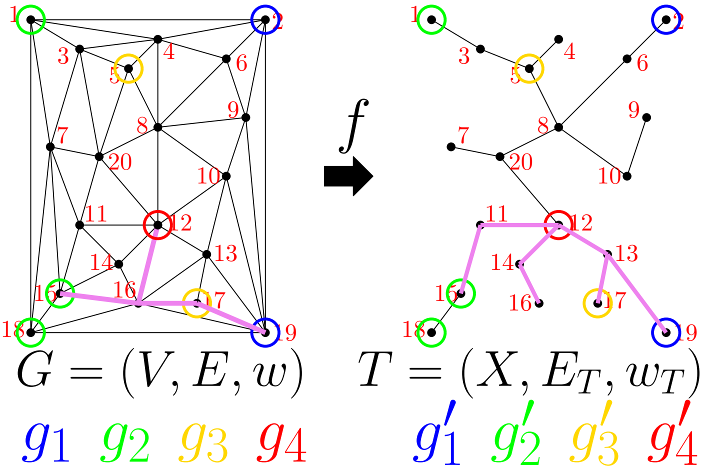
$S^\star_T$: $\forall (u,v) \in S^\star$ add the path from $f(u)$ to $f(v)$.
(valid solution)



GST: Given subsets $g_1, g_2, \dots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i.

Embedding f with expected distortion $O(\log n)$. $\quad g_i' = f(g_i)$

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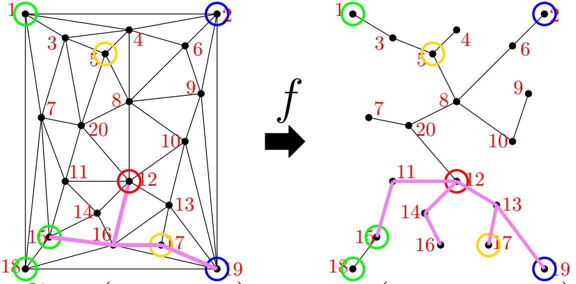

$G = (V, E, w) \qquad T = (X, E_T, w_T)$

$g_1 \quad g_2 \quad g_3 \quad g_4 \qquad g_1' \quad g_2' \quad g_3' \quad g_4'$

$$\mathbb{E}[w_T(S^\star_T)] \le \sum_{(u,v)\in S^\star} \mathbb{E}\left[d_T(f(u), f(v))\right] = O(\log n) \cdot \sum_{(u,v)\in S^\star} d_T(f(u), f(v))$$

$$= O(\log n) \cdot \sum_{(u,v)\in S^\star} w_G((u,v)) = O(\log n) \cdot w_G(S^\star)$$

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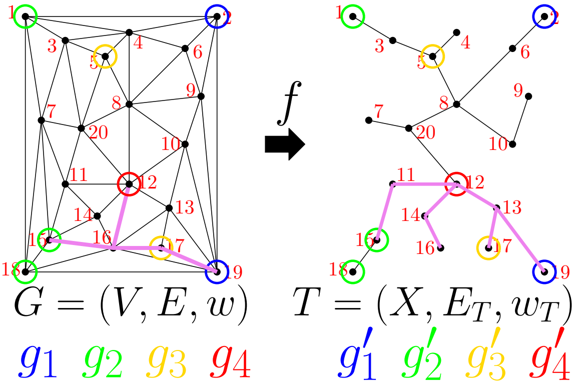

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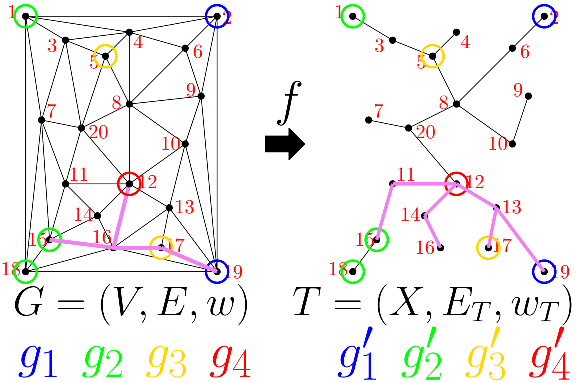

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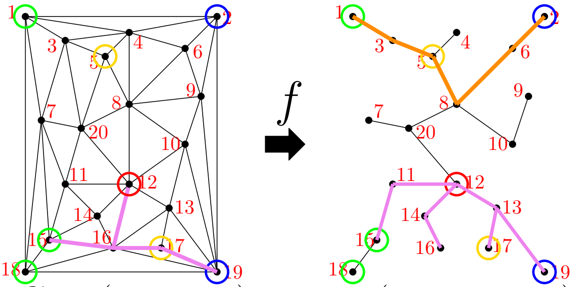

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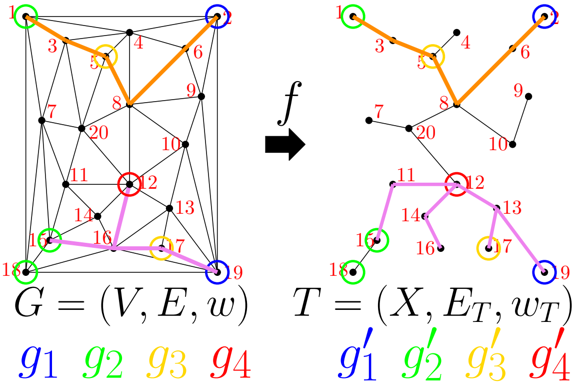

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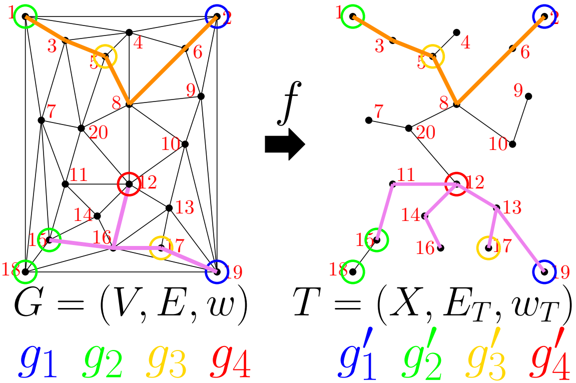

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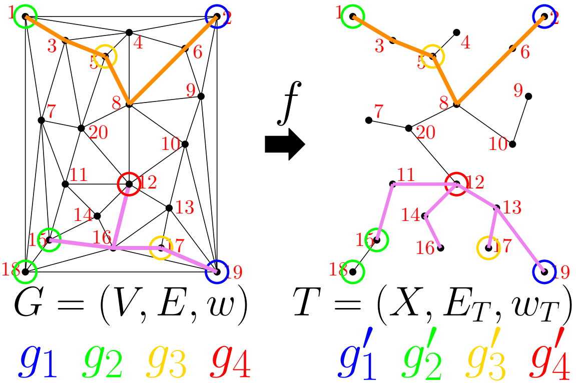

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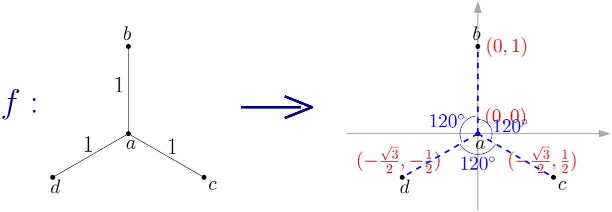
How to get the approximation guarantee with high probability?

Repeat the process $O(\log n)$ times, and return the observed solution of minimum weight.

Outline of the talk

$f : (X, d_X) \to (Y, d_Y)$ has **distortion** t if:

$$\forall x, y \in X, d_X(x, y) \leq d_Y(f(x), f(y)) \leq t \cdot d_X(x, y) \ .$$



	a	b	c	d
a		1	1	1
b	1		2	2
c	1	2		2
d	1	2	2	

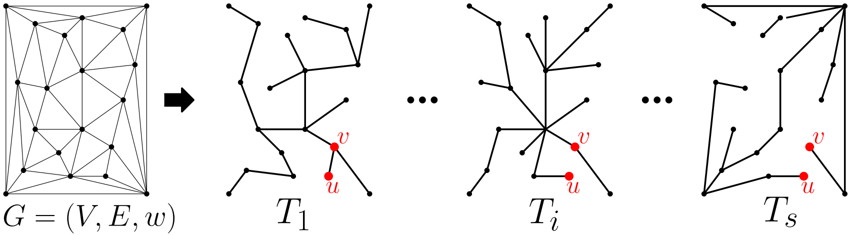
	a	b	c	d
a		$2/\sqrt{3}$	$2/\sqrt{3}$	$2/\sqrt{3}$
b	$2/\sqrt{3}$		2	2
c	$2/\sqrt{3}$	2		2
d	$2/\sqrt{3}$	2	2	

The distortion of the embedding is $\frac{2}{\sqrt{3}} \approx 1.1547$.

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.



For every $u, v \in X$ and $T \in \mathrm{supp}(\mathcal{D})$, $d_X(u, v) \le d_T(f(u), f(v))$.

For every $u, v \in X$ $\quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(f(u), f(v))] \le O(\log n) \cdot d_X(u, v)$.

[Alon, Karp, Peleg, West 95]: Tight!

Distance Oracle construction

Sample $s = 4\log n$ trees $T_1, \ldots, T_s$. Given x, y return $\mathrm{DO}(x,y) = \min_{i\in[1,s]} d_{T_i}(x,y)$.

Clearly, as the trees are **dominating**, $\mathrm{DO}(x,y) = \min_{i\in[1,s]} d_{T_i}(x,y) \geq d_X(x,y)$.

Thus with high probability, for every $x, y \in X$

$$\mathrm{DO}(x,y) < 2 \cdot \mathbb{E}_{T\sim\mathcal{D}}[d_T(x,y)] = O(\log n) \cdot d_G(x,y) \ .$$

Space: storing $O(\log n)$ trees. Total space is $O(n\log n)$ (machine words).

Query time: computing $d_{T_i}(x,y)$ for $i \in [1,s]$.

There is a data structure computing distance in (some) trees in $O(1)$ time.

Overall $O(\log n)$ query time.

Overall we obtained distance approximation $O(\log n)$ with $O(\log n)$ query time and $O(n\log n)$ space.

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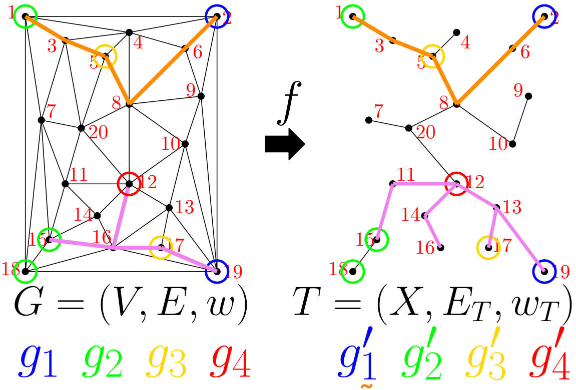

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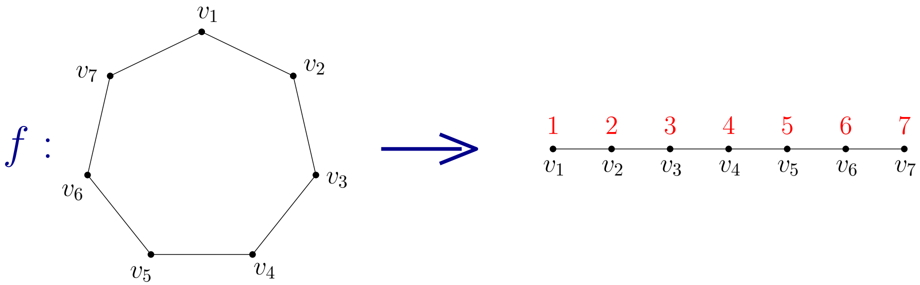
You can learn about my research from many different videos in my home-page.

Quiz.

Q0: Consider an embedding of the circle graph C_7 into the line, such that the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ are mapped to $\{1, 2, 3, 4, 5, 6, 7\}$ respectively.
What is the distortion?

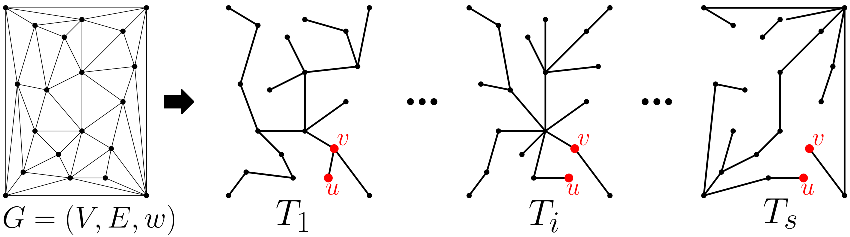


Quiz. Consider a graph family called *Graphica Prime*, such that every graph G in the family embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** t.

Stochastic Embedding into Trees

Theorem (Stochastic embedding for Graphica Prime)

*Every graph $G = (V, E, w)$ in Graphica Prime embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $\boldsymbol{t}$.*



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Quiz.

Consider a graph family called *Graphica Prime*, such that every n-graph G in the family embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** t. Using similar techniques to what we did in class:

Q1: What **distance oracle** can you achieve for graphs in *Graphica Prime*?

Q2: What **approximation factor** can you obtain for graphs in *Graphica Prime* for the group Steiner tree problem?

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Link to quiz:

Can also be found in my homepage:

arnold.filtser.com

Or just google Arnold Filtser.

Link to slides:

Outline of the talk - Appendix

We will begin our tour of metric embeddings into trees with the classics: [Bartal 96]

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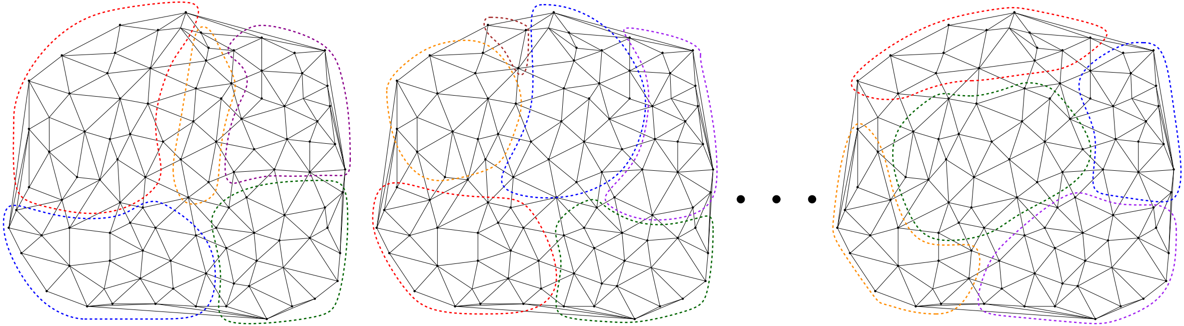
This one is based on random partitions of metric spaces.

Definition (Padded Decomposition)

Given a metric space (X, d_X) (or a weight graph $G = (V, E, w)$).
Distribution $\mathcal{D}$ over partitions of G is (β, Δ)-padded decomposition if:



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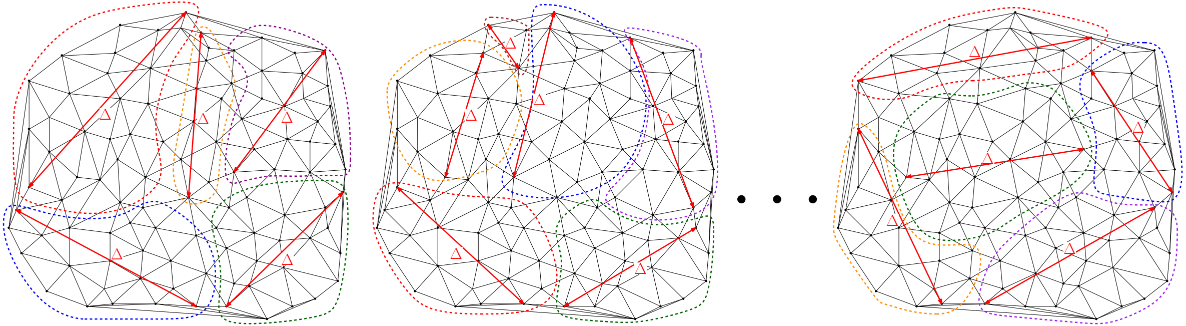

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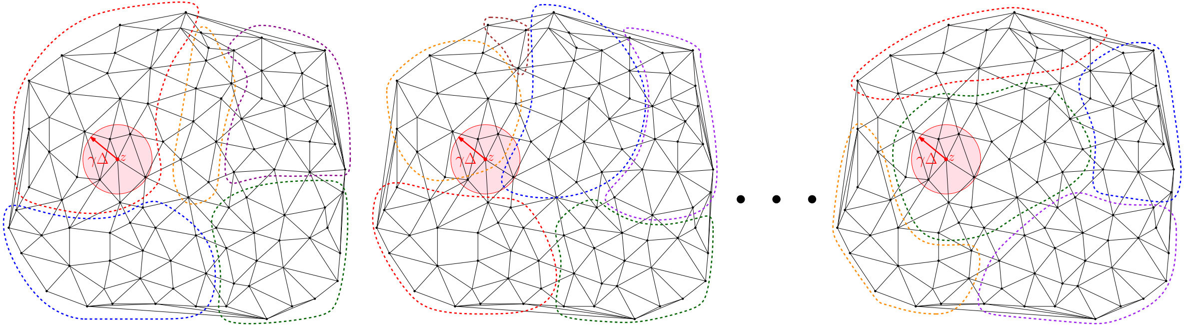

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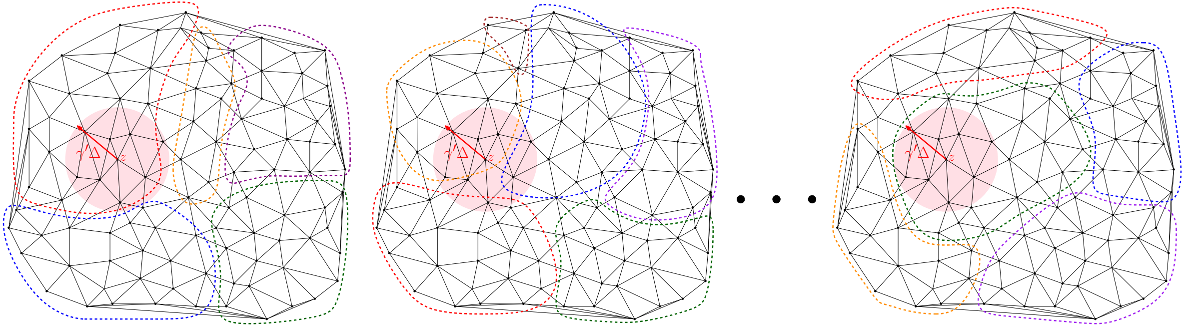

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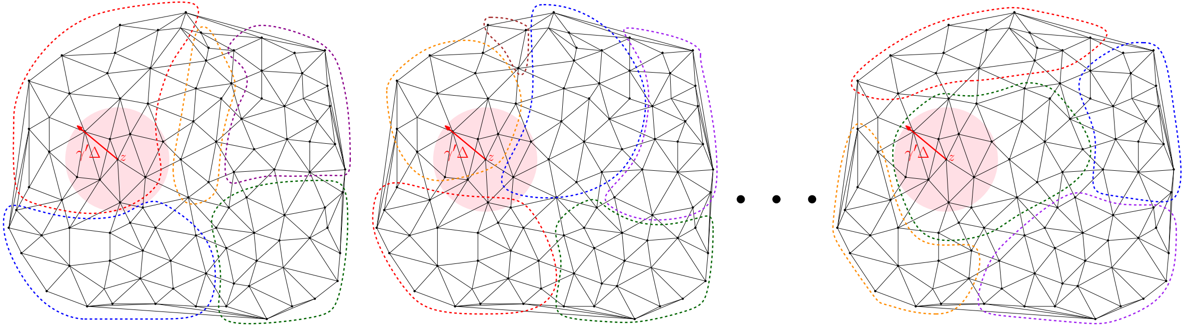

G admits a β-padded decomposition **scheme**:

$\forall \Delta > 0$, G admits (β, Δ)-padded decomposition.

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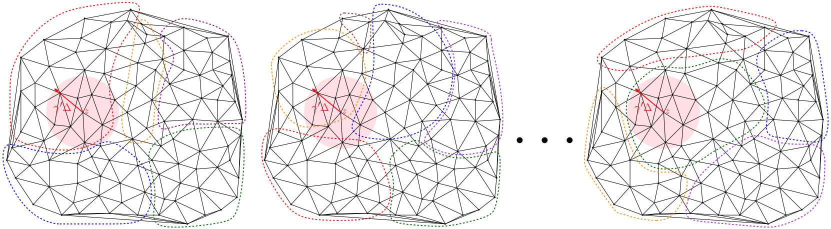

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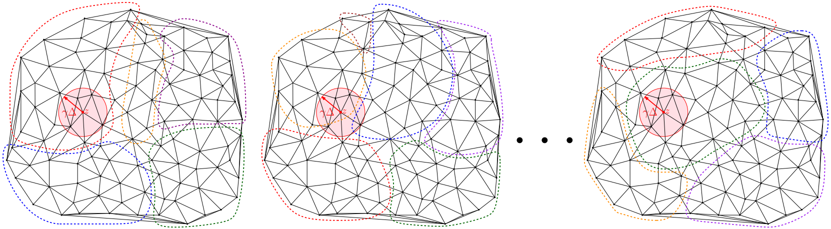

Note: $\Pr[B(z, \frac{1}{\beta} \cdot \Delta) \subseteq P(z)] \geq \Omega(1)$.

For small enough γ, cut probability: $\Pr[B(z, \gamma\Delta) \not\subseteq P(z)] \leq 1 - e^{-\beta\gamma} \approx \beta\gamma$.

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This is also tight! [Bartal 96]

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Algorithm:

1. Arbitrarily order X: $x_1, x_2, \ldots, x_n$.
2. For $i = 1$ to n
   1. Sample $r_i \sim \mathrm{Exp}(1)$.
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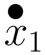

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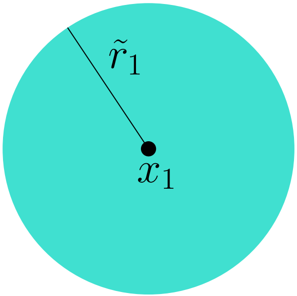

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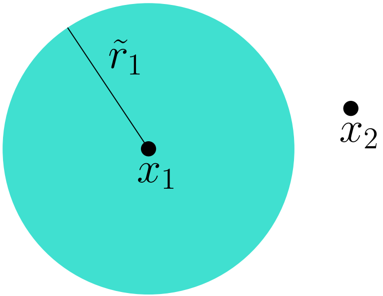

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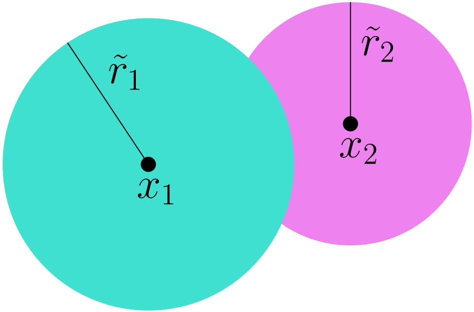

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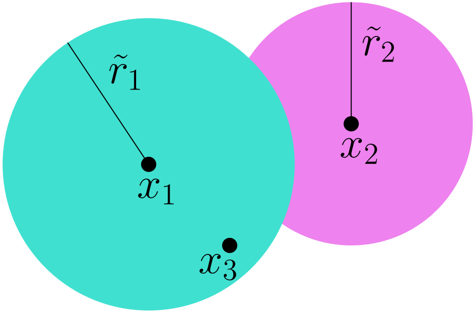

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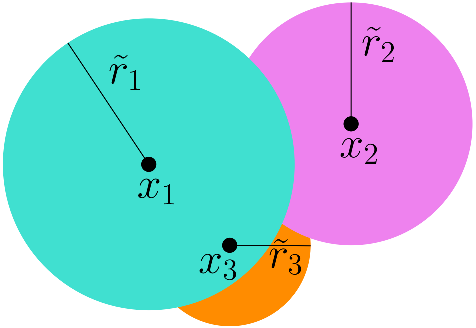

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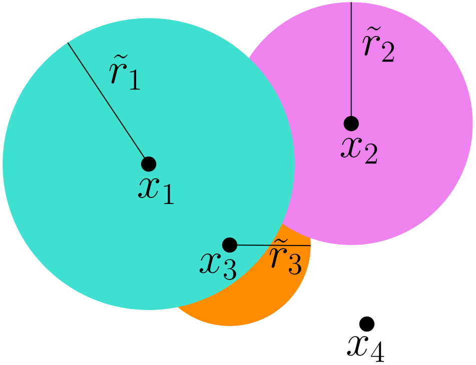

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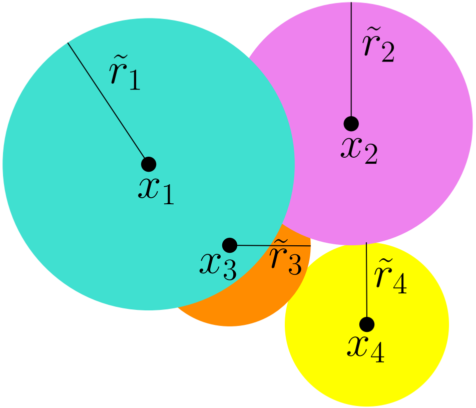

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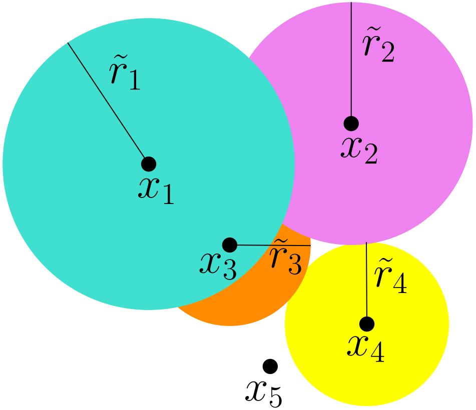

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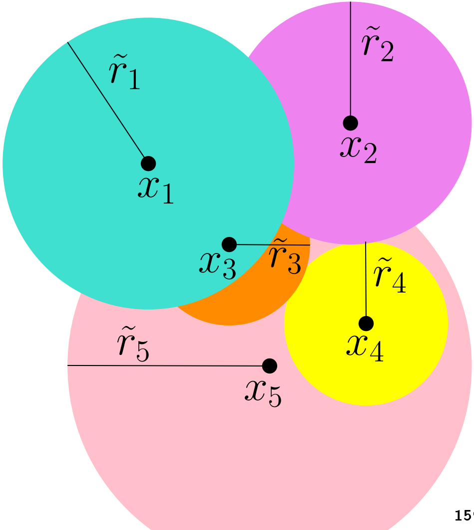

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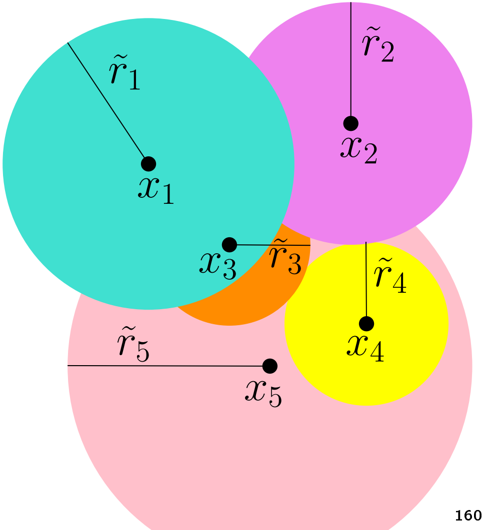

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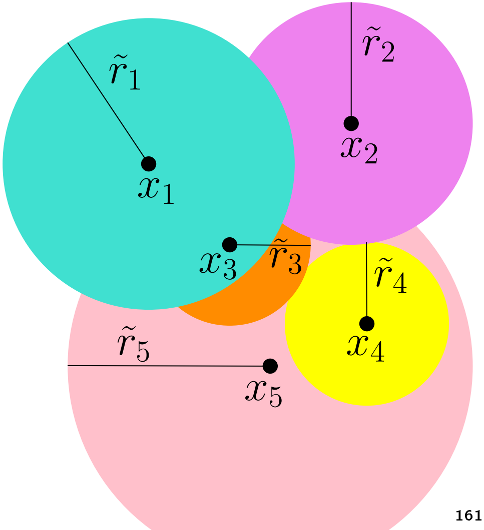

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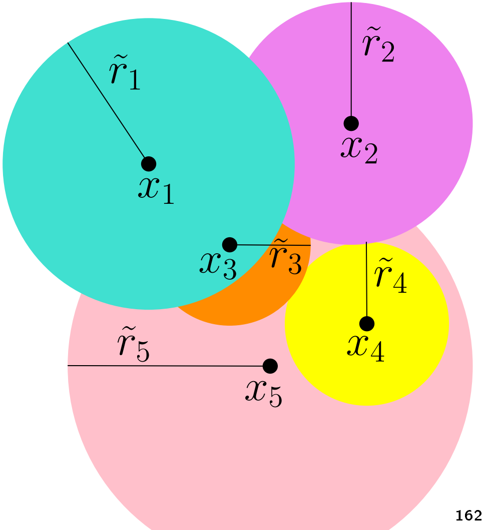
$\Rightarrow$ all clusters have diameter $\leq \Delta$.



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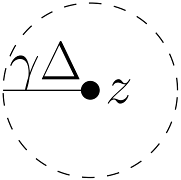

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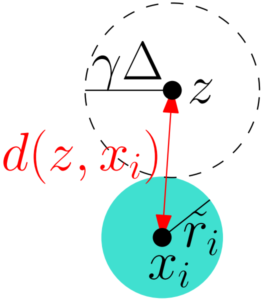

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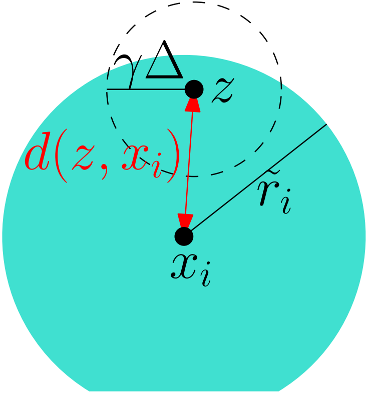

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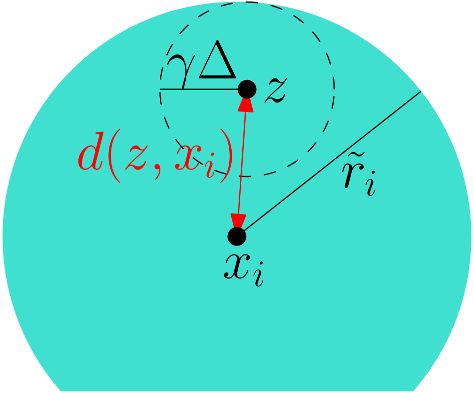

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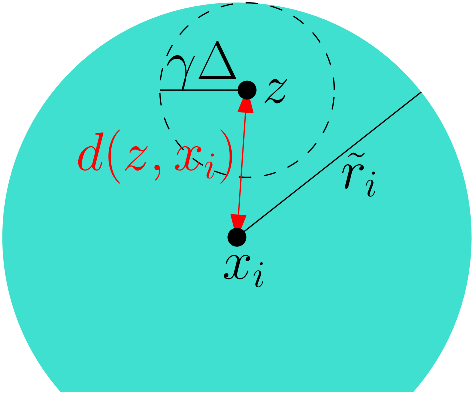

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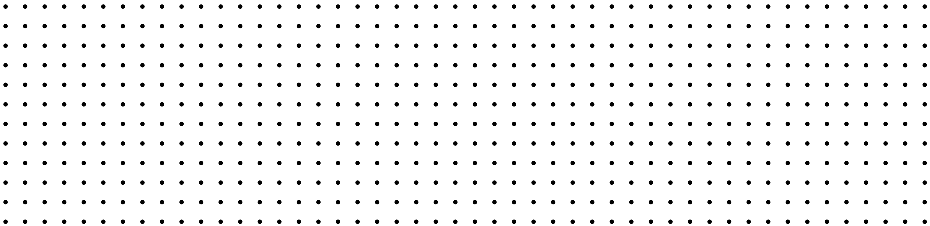

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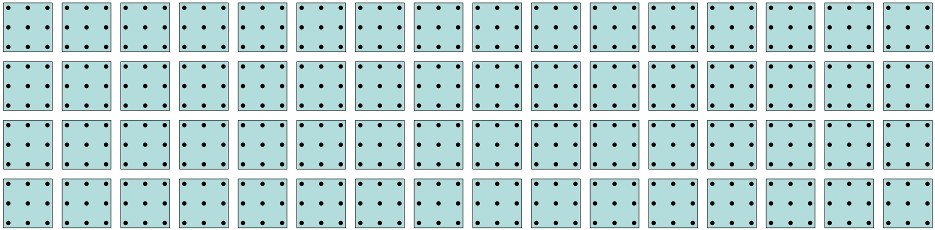

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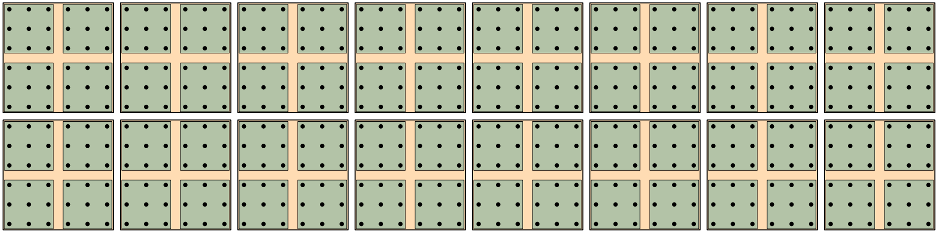

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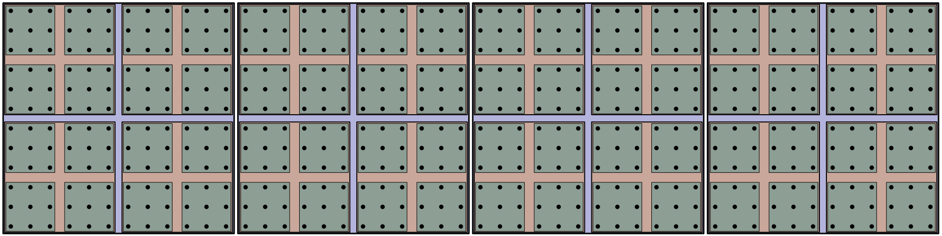

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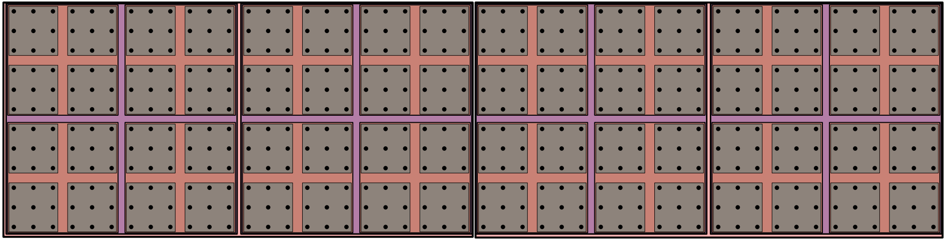

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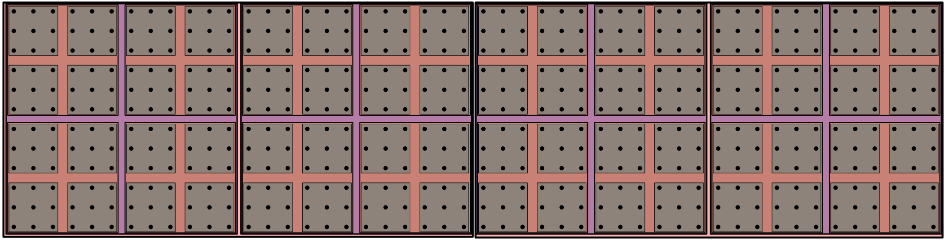

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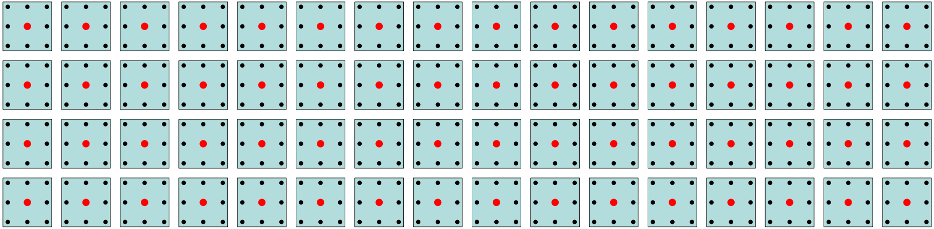

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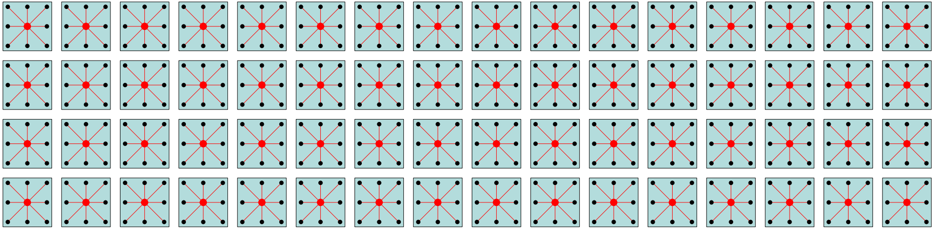

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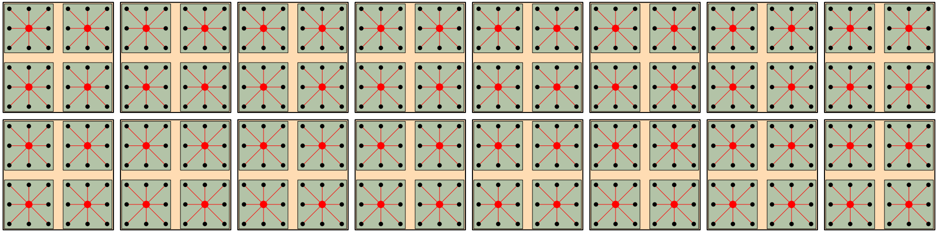

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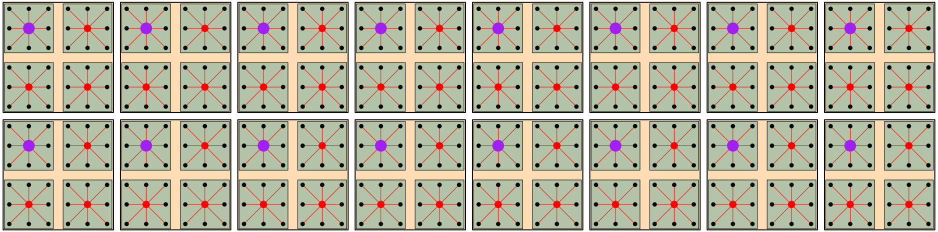

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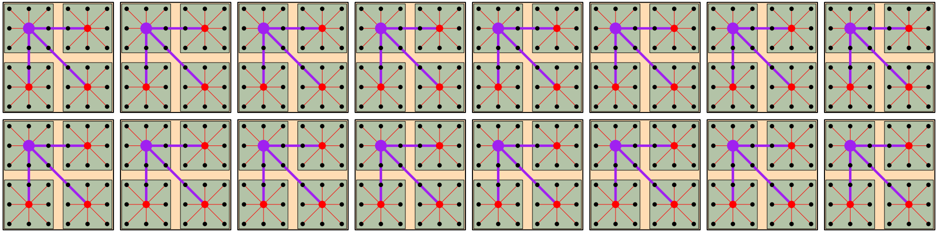

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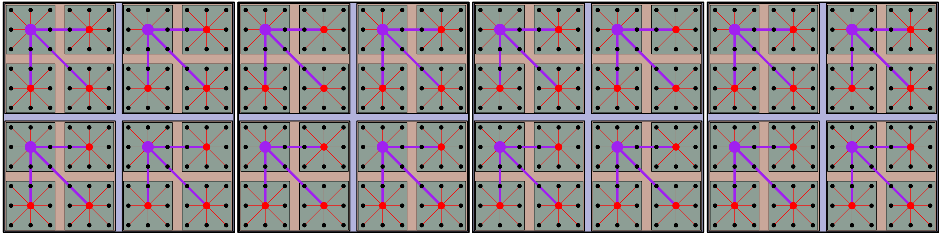

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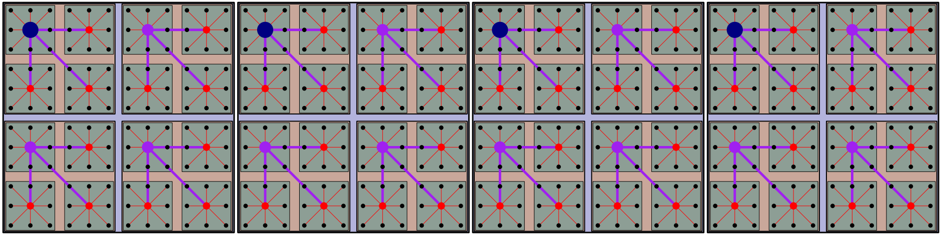

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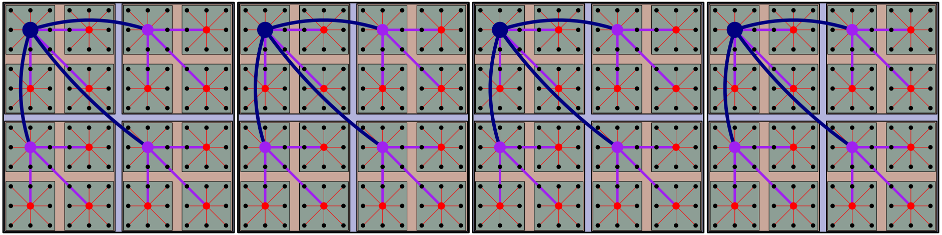

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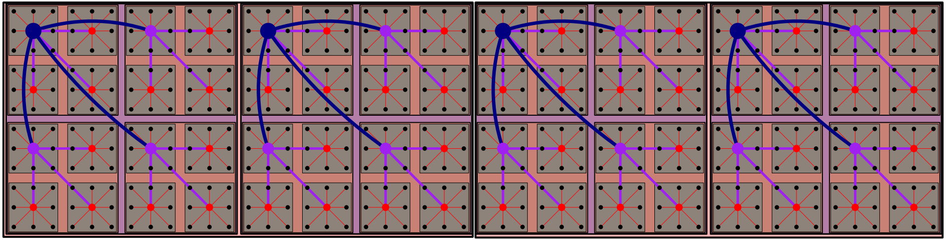

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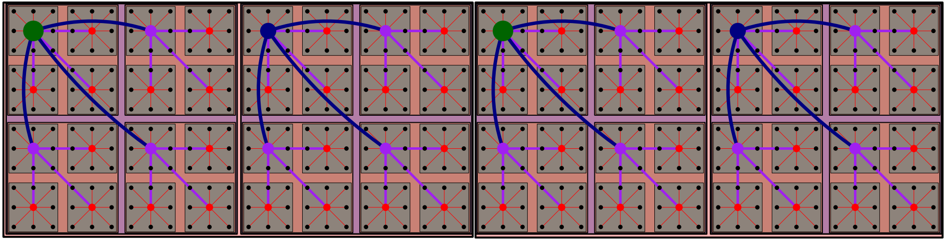

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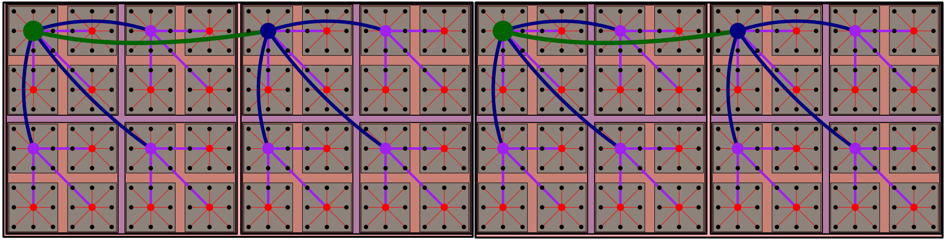

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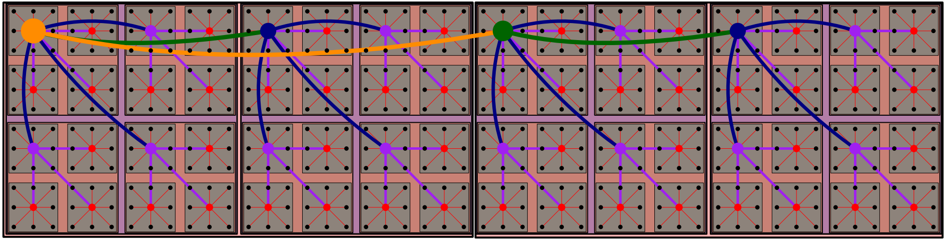

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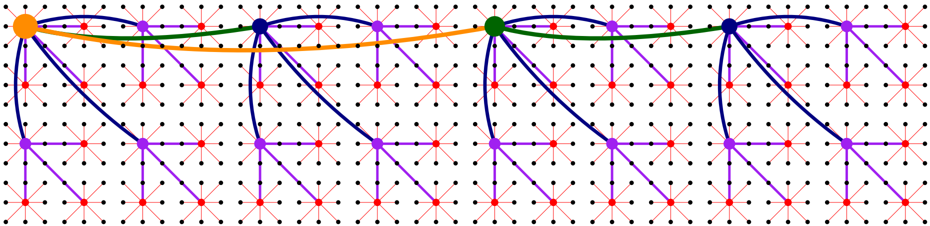

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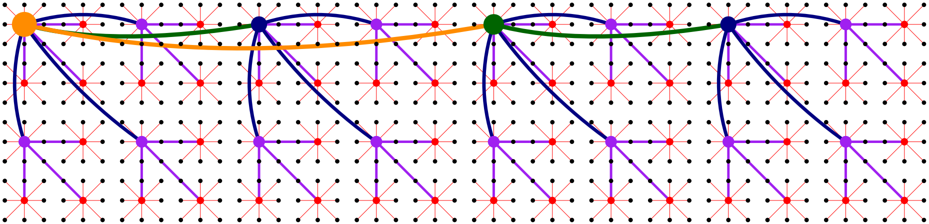

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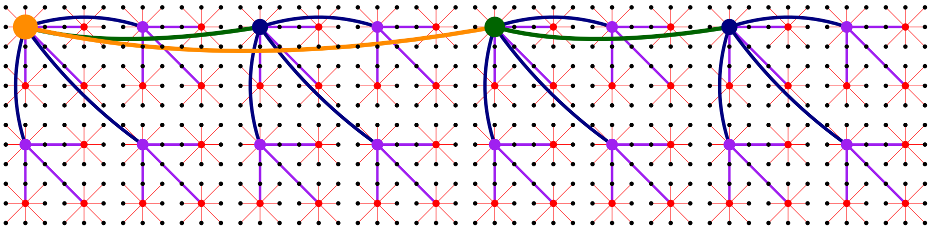

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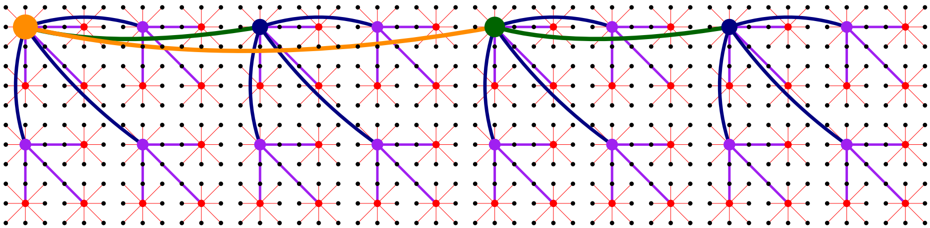

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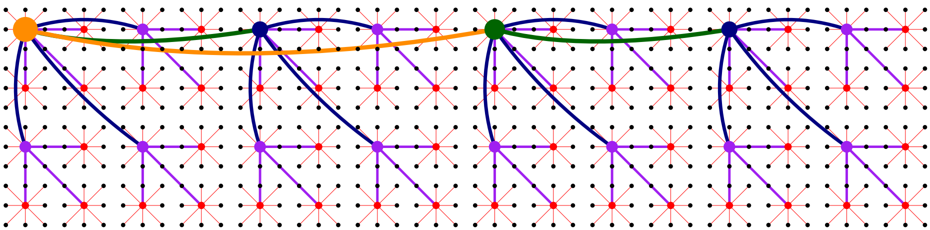

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Specifically, the probability to cut x, y at scale Δ is

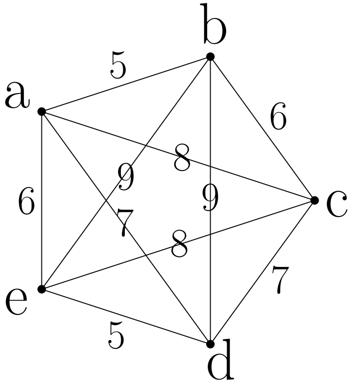
$$\approx \frac{d_X(x, y)}{\Delta} \cdot \log \frac{|B(x, c \cdot 2^i)|}{|B(x, 2^i/c)|}$$

for some constant c, instead of $\approx \frac{d_X(x,y)}{\Delta} \cdot \log n$. Then the sum "telescopes".

Outline of the talk - Appendix

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.



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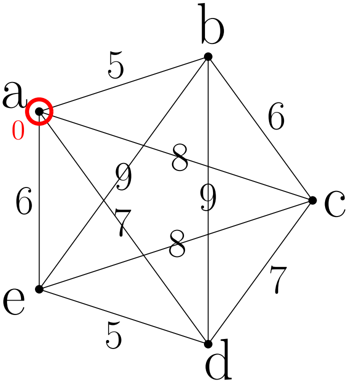

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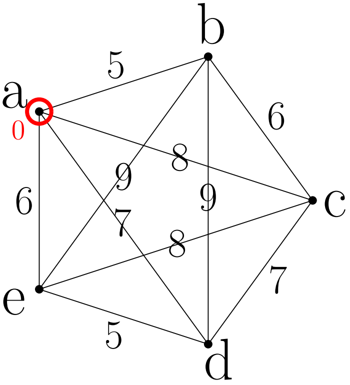

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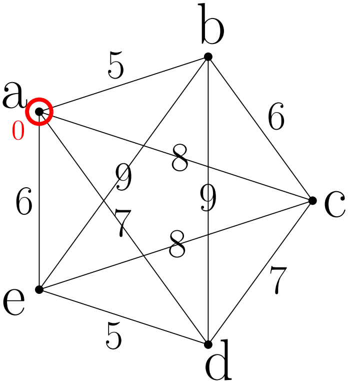

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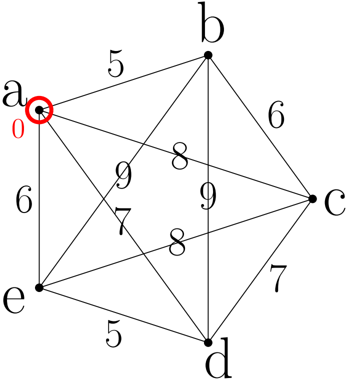

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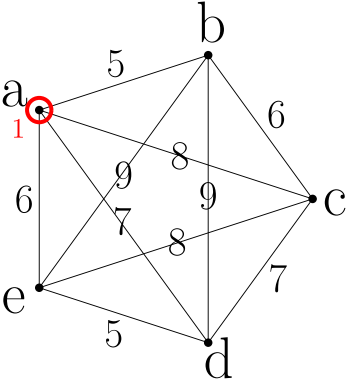

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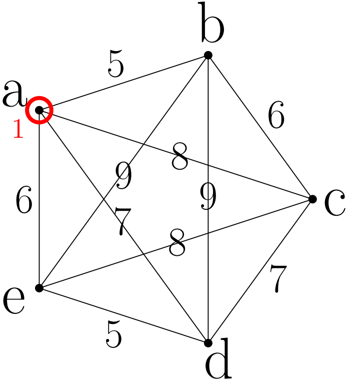
Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_1(a) = 3$

$f_1(b) = 1$

$f_1(c) = 2$

$f_1(d) = 4$

$f_1(e) = 1$

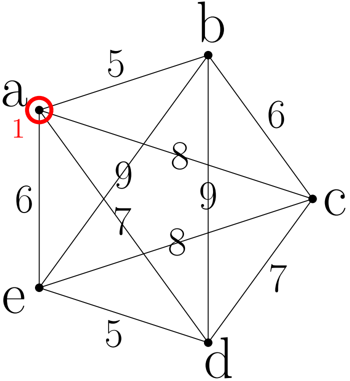
Cost alg(1)=3

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

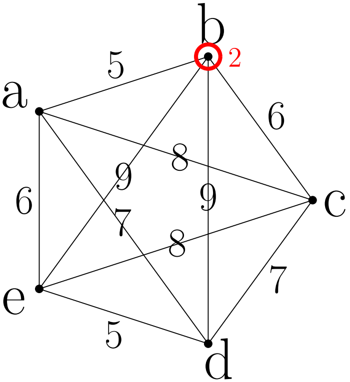
Cost alg(1)=3

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

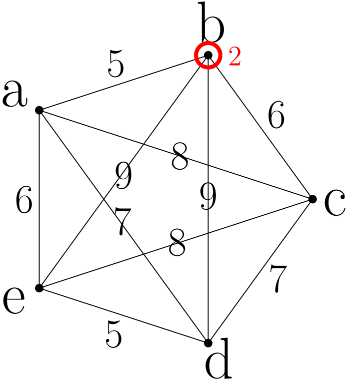
Cost alg(1)=3

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

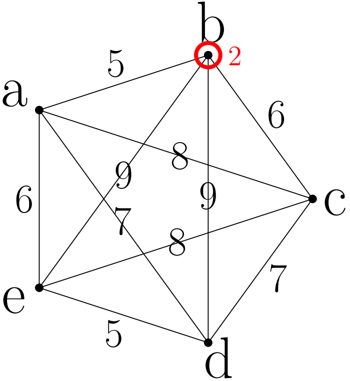
Cost alg(2)$=3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

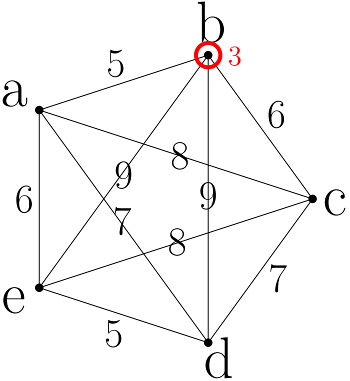
Cost alg(2)$=3+5+1=9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

Cost alg(2)$=3 + 5 + 1 = 9$

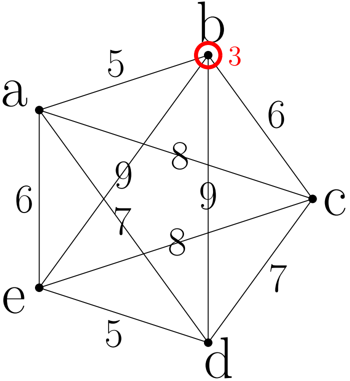
Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.

At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

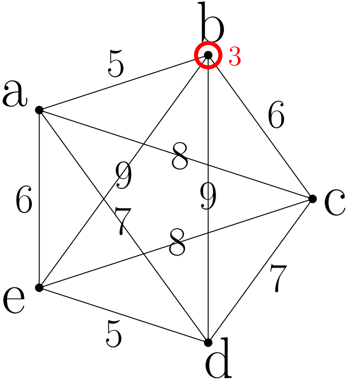
Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

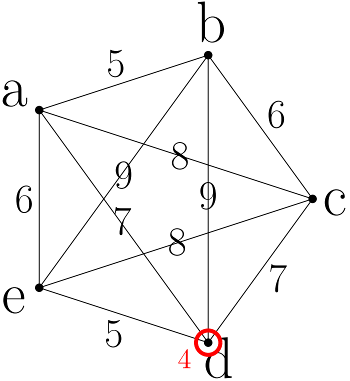
Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

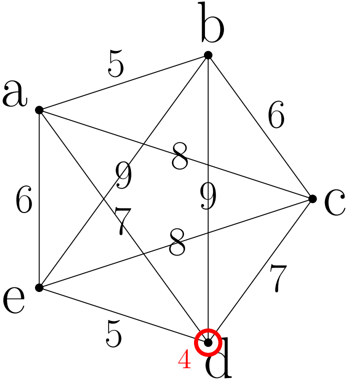


$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.
Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

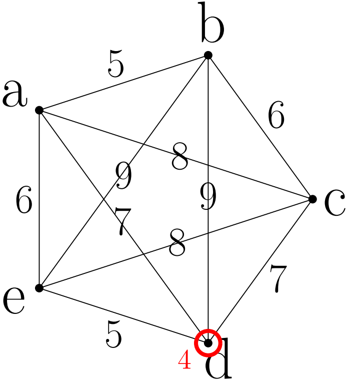
Cost alg(4)$=13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

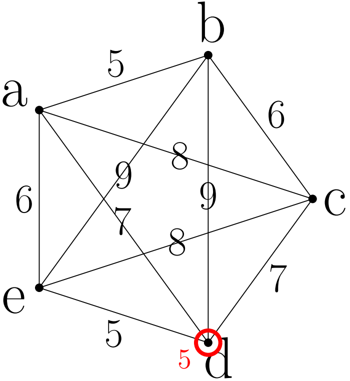
Cost alg(4)=$13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

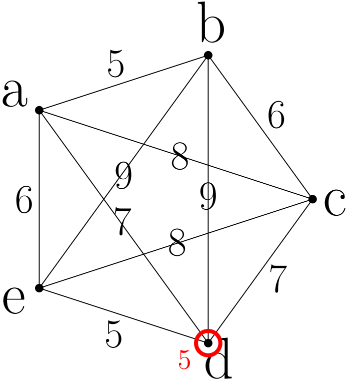
Cost alg(4)=$13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

Cost alg(5)=22 + 5 = 27

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

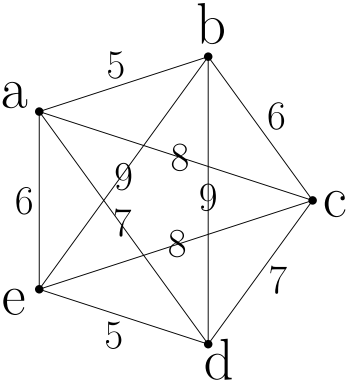
What about Opt?

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

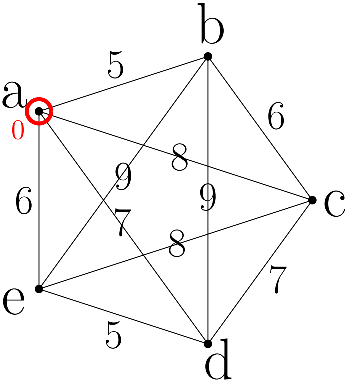


Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

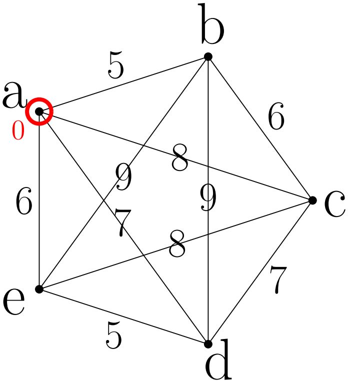


Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



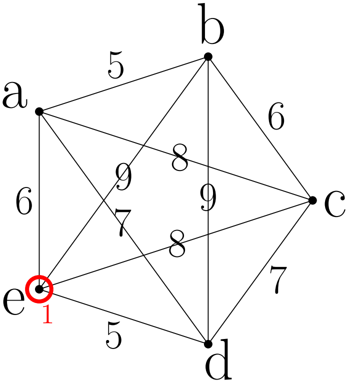
$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



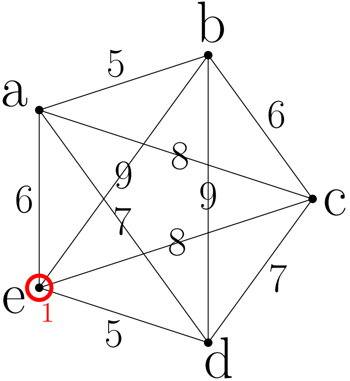
$f_1(a) = 3$
$f_1(b) = 1$
$f_1(c) = 2$
$f_1(d) = 4$
$f_1(e) = 1$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

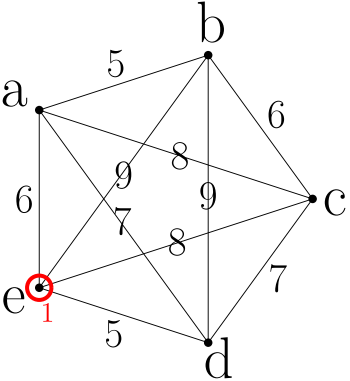
Cost opt(1)$=6 + 1 = 7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_2(a) = 3$

$f_2(b) = 1$

$f_2(c) = 3$

$f_2(d) = 2$

$f_2(e) = 1$

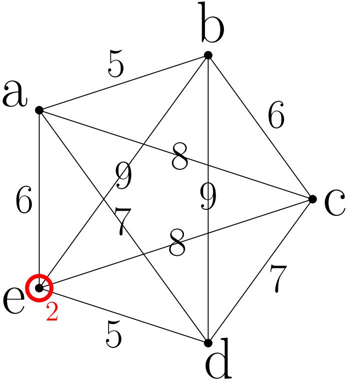
Cost opt(1)$=6+1=7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$

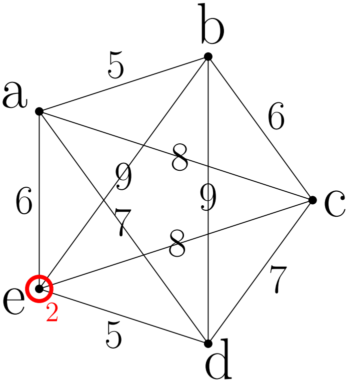
Cost opt(1)$=6 + 1 = 7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

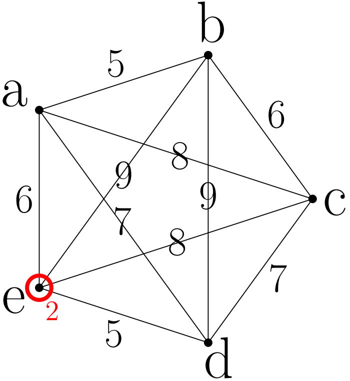
Cost opt(2)$=7 + 1 = 8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

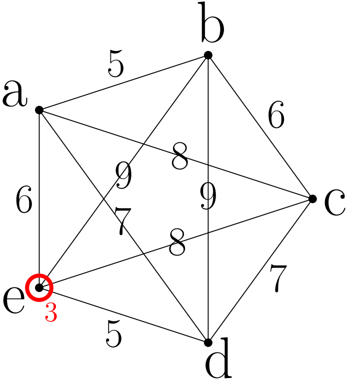
Cost opt(2)$=7 + 1 = 8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

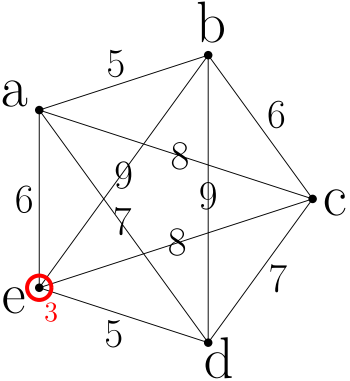
Cost opt(2)$=7 + 1 = 8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

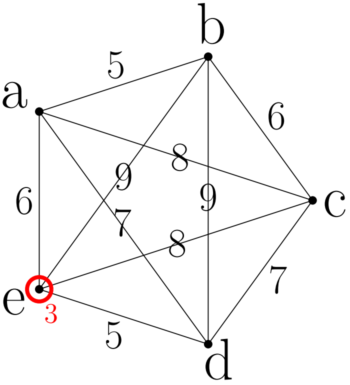
Cost opt(3)=$8 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

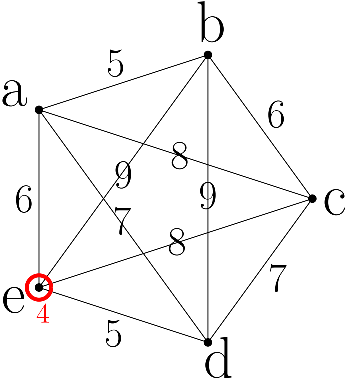
Cost opt(3)$=8+1=9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
$$f_4(e) = 1$$

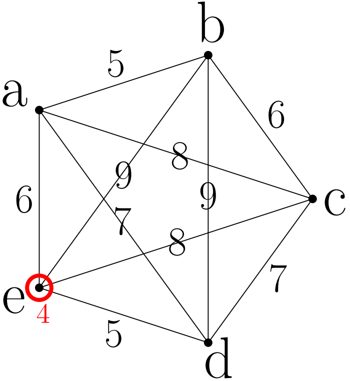
Cost opt(3)$=8+1=9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

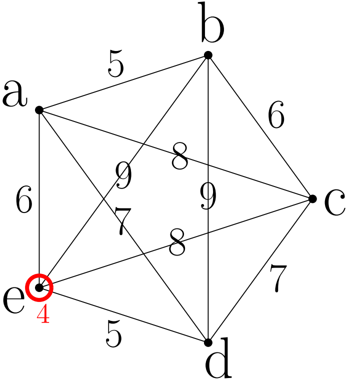
Cost opt(4)$=9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

Cost opt(4)=$9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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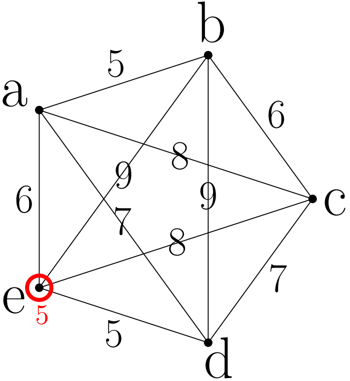
Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

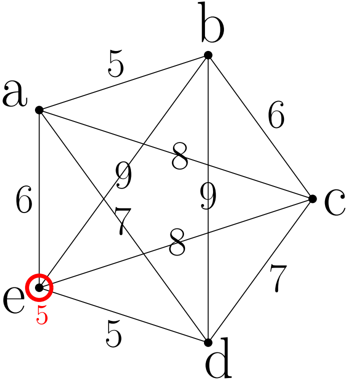
Cost opt(4)=$9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

Cost opt(5)=$10 + 1 = 11$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Here $\frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)} = \frac{27}{11}$.

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.
Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Here $\frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)} = \frac{27}{11}$.

Competitive ratio against oblivious adversary is

$$\max_{\text{input } I} \frac{\mathbb{E}[\mathrm{Alg}(I)]}{\mathrm{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log \log n)$ against oblivious adversary.*

* Actually on an HST, which is a special kind of tree. [FRT04] is into HST's.

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree* T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log\log n)$ against oblivious adversary.

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree * T*, there is an online algorithm for MTS with competitive ratio* $O(\log n \cdot \log\log n)$ *against oblivious adversary.*

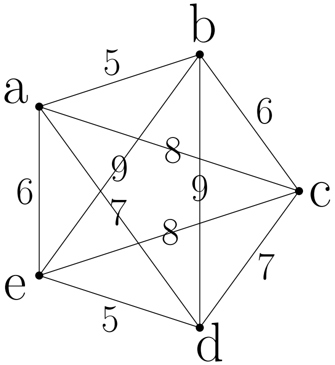
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

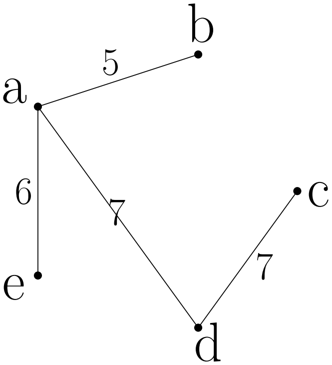
2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].





Online problem - Metrical Task System (MTS)

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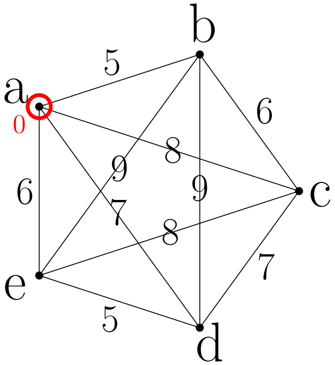
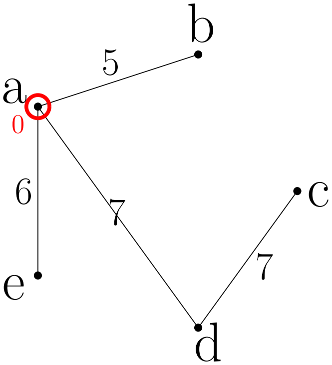
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].





Online problem - Metrical Task System (MTS)

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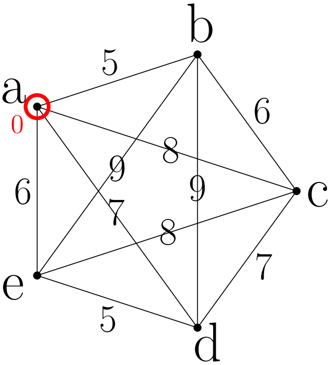
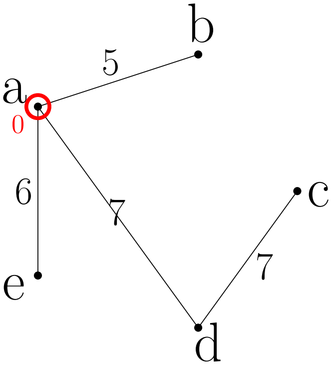


$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$



Online problem - Metrical Task System (MTS)

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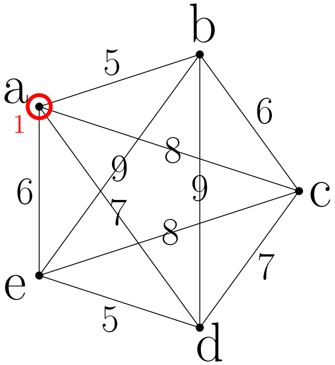
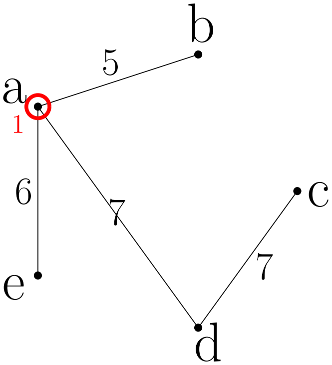
Make the same decisions as [FM00].



$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$



Online problem - Metrical Task System (MTS)

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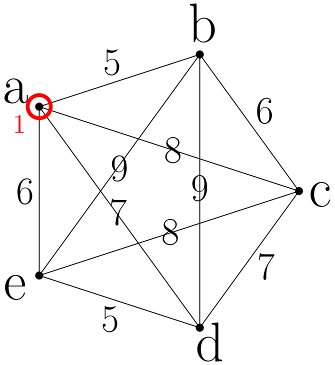
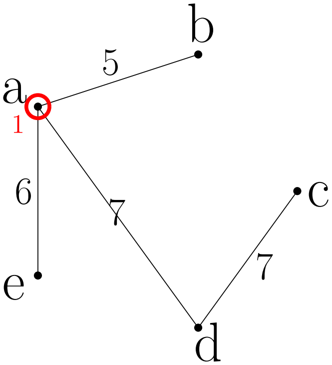


$f_1(a) = 3$
$f_1(b) = 1$
$f_1(c) = 2$
$f_1(d) = 4$
$f_1(e) = 1$



Cost alg(1)=3

Cost $\text{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
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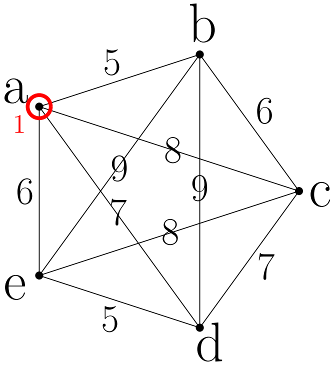
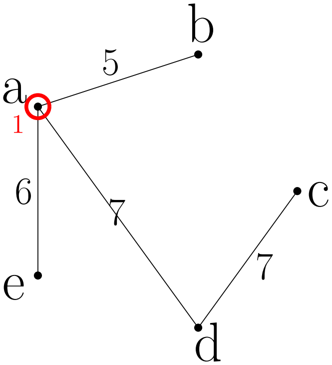


Cost alg(1)=3

$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$



Cost $\text{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
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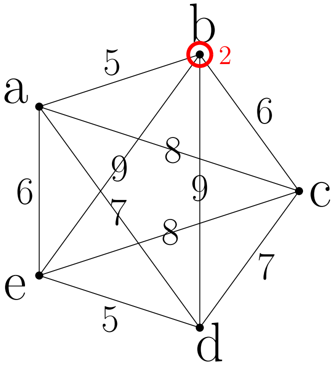
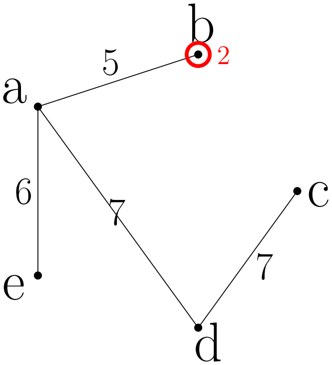


$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$



Cost alg(1)=3

Cost $\text{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

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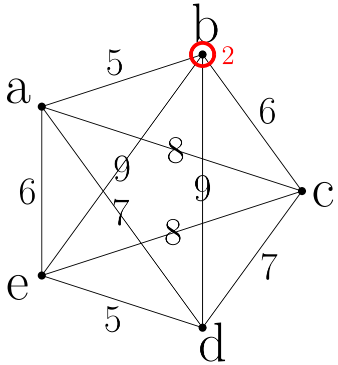
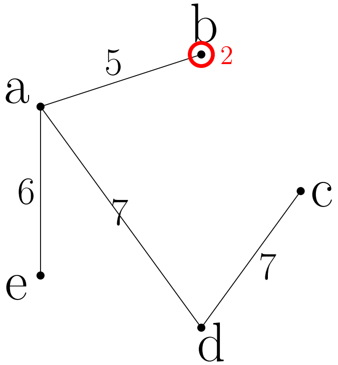
Make the same decisions as [FM00].



$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$



Cost alg(2)=$3 + 5 + 1 = 9$

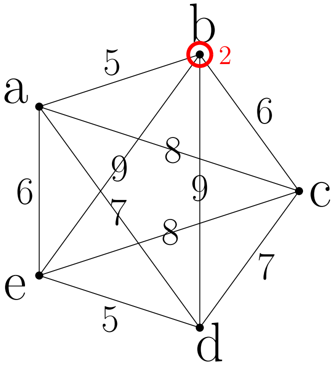
Cost $\text{alg}_T(2)$=$3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

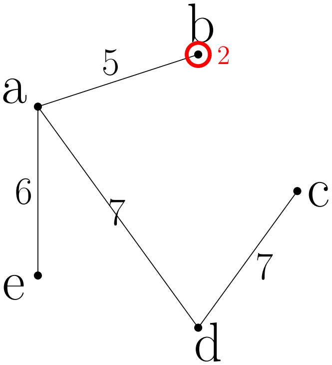
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$



Cost alg(2)$=3+5+1=9$

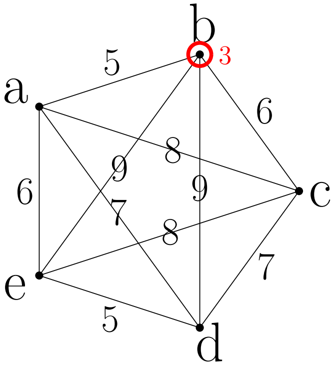
Cost $\text{alg}_T(2)$$=3+5+1=9$

Online problem - Metrical Task System (MTS)

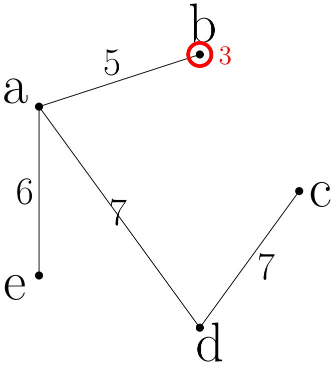
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

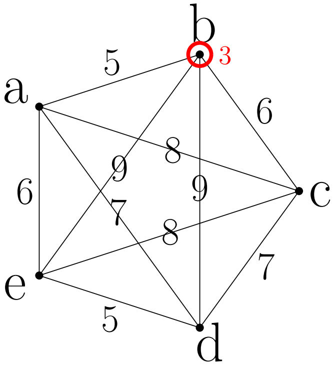


Cost alg(2)=$3 + 5 + 1 = 9$

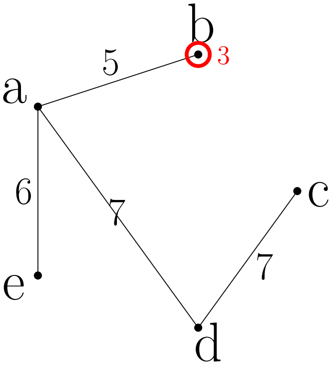
Cost $\text{alg}_T(2)$=$3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$



Cost alg(3)=$9 + 4 = 13$

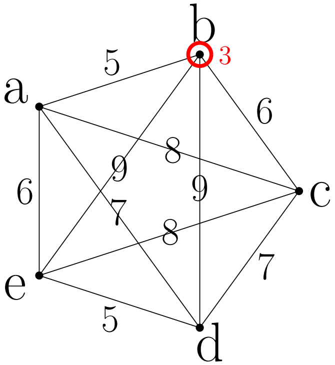
Cost $\mathrm{alg}_T(3)$=$9 + 4 = 13$

Online problem - Metrical Task System (MTS)

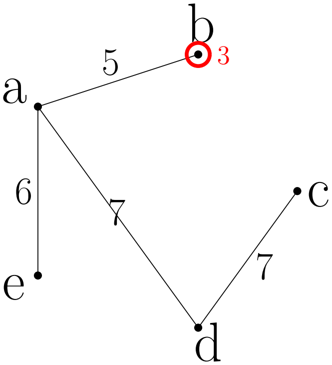
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
$$f_4(e) = 1$$

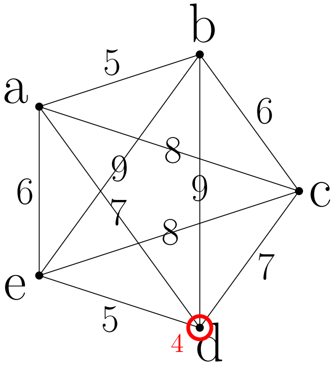


Cost $\mathrm{alg}(3) = 9 + 4 = 13$

Cost $\mathrm{alg}_T(3) = 9 + 4 = 13$

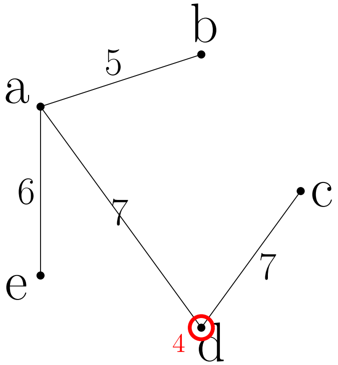
Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].



Cost alg(3)=$9+4=13$

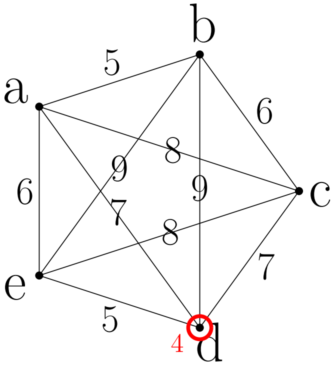
$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$



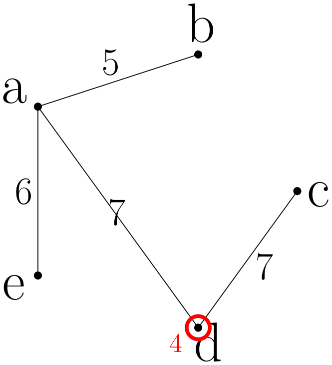
Cost $\mathrm{alg}_T(3)$=$9+4=13$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].



$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
$$f_4(e) = 1$$



Cost $\text{alg}(4) = 13 + 9 + 0 = 22$

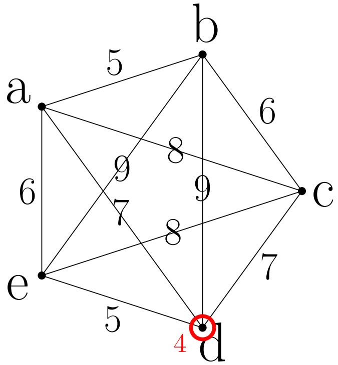
Cost $\text{alg}_T(4) = 13 + 12 + 0 = 25$

Online problem - Metrical Task System (MTS)

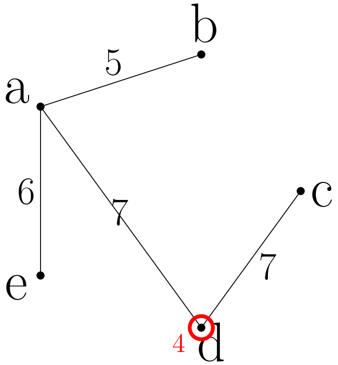
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$



Cost alg(4)=$13 + 9 + 0 = 22$

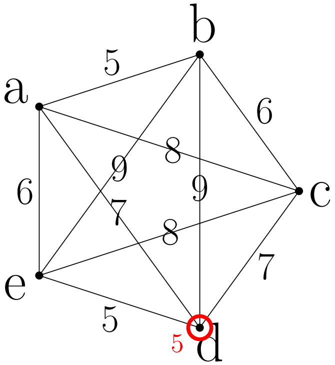
Cost $\text{alg}_T(4)$=$13 + 12 + 0 = 25$

Online problem - Metrical Task System (MTS)

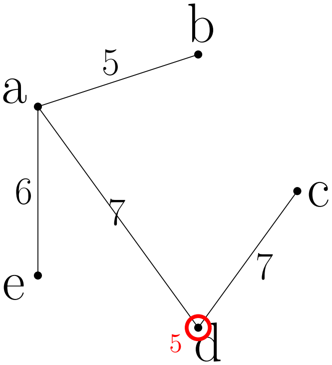
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$



Cost $\mathrm{alg}(4) = 13 + 9 + 0 = 22$

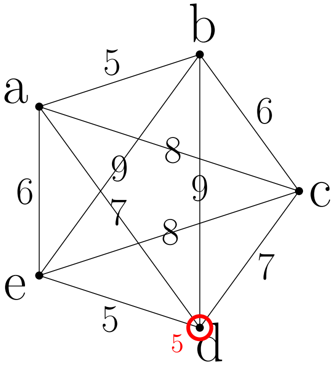
Cost $\mathrm{alg}_T(4) = 13 + 12 + 0 = 25$

Online problem - Metrical Task System (MTS)

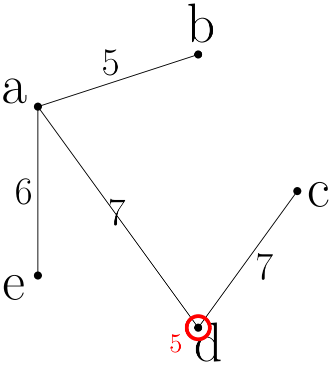
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$



Cost alg(5)=$22 + 5 = 27$

Cost alg$_T(5)$=$25 + 5 = 30$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].

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[FM00] is $O(\log n \cdot \log\log n)$-competitive on T. Hence it choose points $y_1, \ldots, y_k$ such that $\mathbb{E}[\mathrm{alg}_T] = \mathbb{E}[\sum_{i=1}^{k} f_i(y_i) + \sum_{i=1}^{k} d_T(y_{i-1}, y_i)] \leq O(\log n \cdot \log\log n) \cdot \mathrm{opt}_T$.

We made the same decisions, so overall:

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Theorem

MTS has an $O(\log^2 n \cdot \log\log n)$ competitive algorithm against oblivious adversary.

Outline of the talk - Appendix

Ramsey type Embeddings

Ramsey type theorem: Every **big** enough object, contains a **structured subset**.

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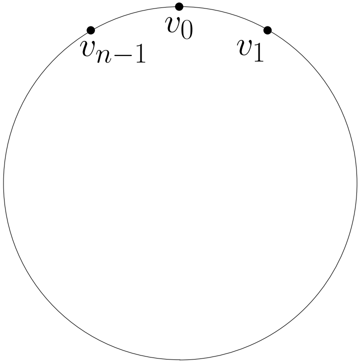

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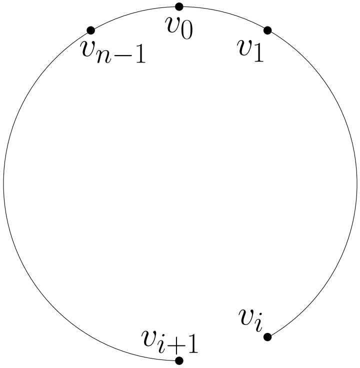

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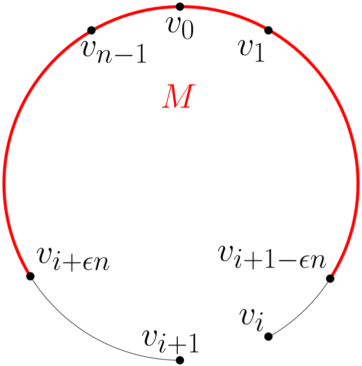

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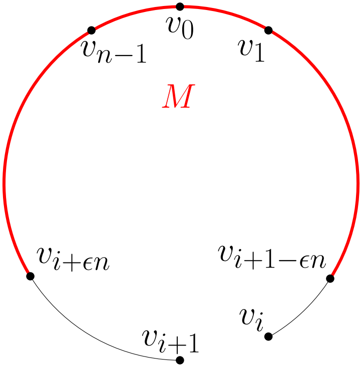

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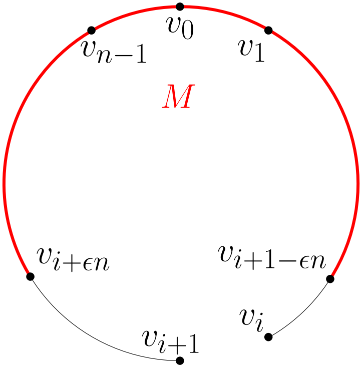

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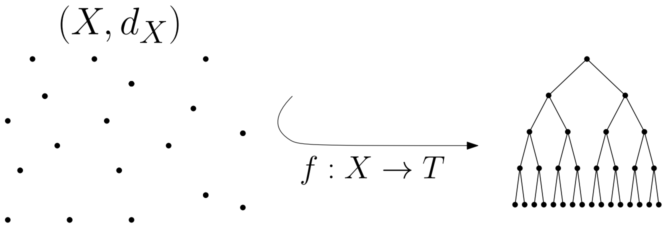
Choose i u.a.r., then $\Pr[v \in M] = 1 - 2\epsilon.$

Ramsey type Embeddings

Fix $k > 1$, what is the **largest subset** $M \subset X$,
s.t. (M, d_X) embeds into a tree with **distortion** k?



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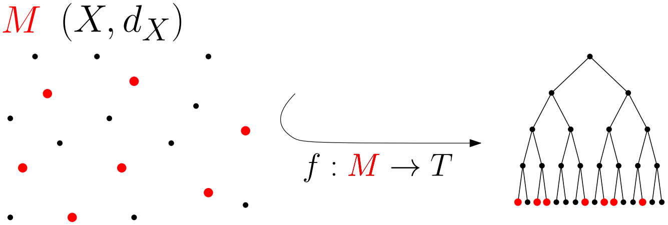

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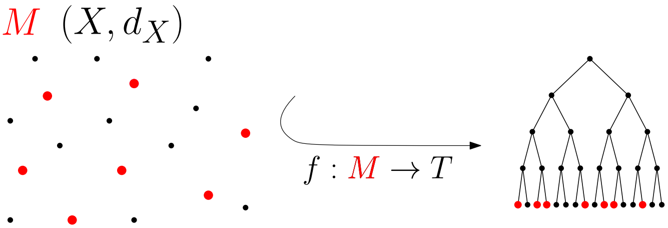

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Theorem ([Mendel, Naor 07], following [BFM86, BLMN05])

*$\forall$ n-point metric space and $k \ge 1$, $\exists$ **subset** M of size $n^{1-1/k}$ that **embeds** into a **tree** with distortion $O(k)$.*

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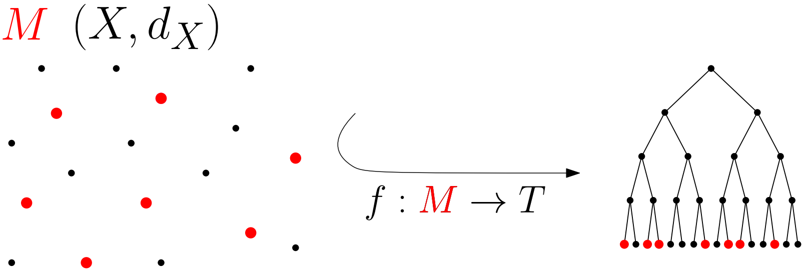

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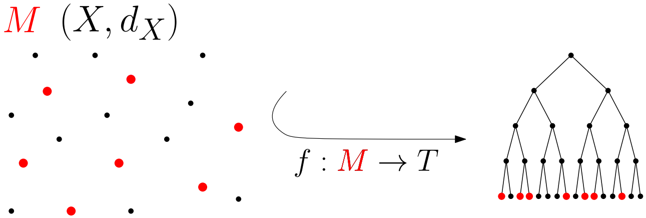

Asymptotically tight.

[Naor, Tao 12]: distortion $2e \cdot k$.

Ramsey type Embeddings

Corollary

For every n-point metric space and $\boldsymbol{k} \geq 1$, *there is a* ***set*** $\mathcal{T}$ *of* $k \cdot n^{\frac{1}{k}}$ ***trees*** *and a* *mapping* **home** : $X \to \mathcal{T}$, *such that for every* $x, y \in X$,

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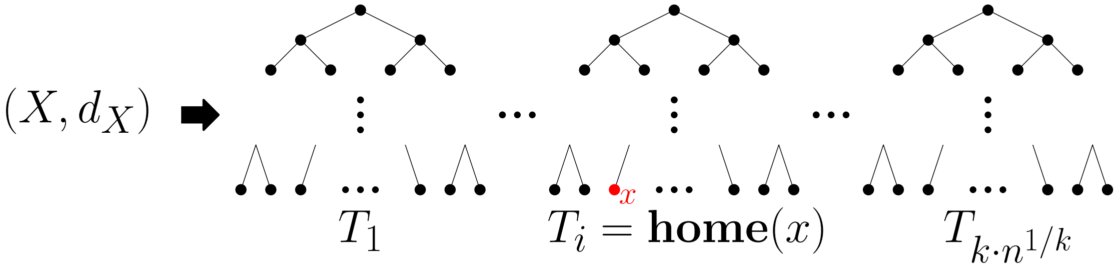

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Applications:

- **Distance oracle**
- Compact routing scheme
- Online algorithms
- Approximate ranking
- etc.

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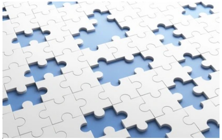
Compromises: only partial guarantees



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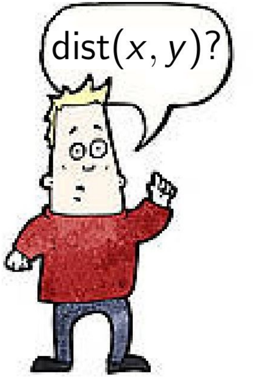

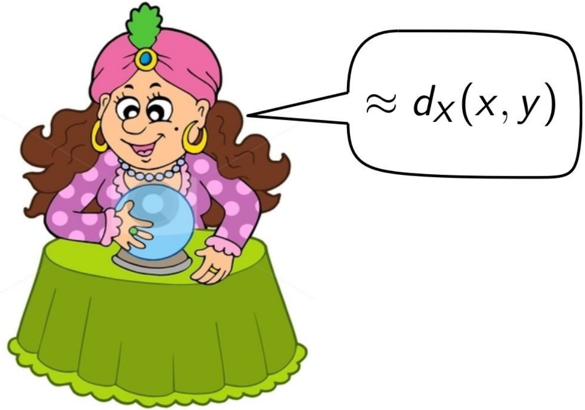

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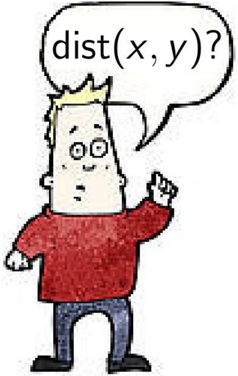

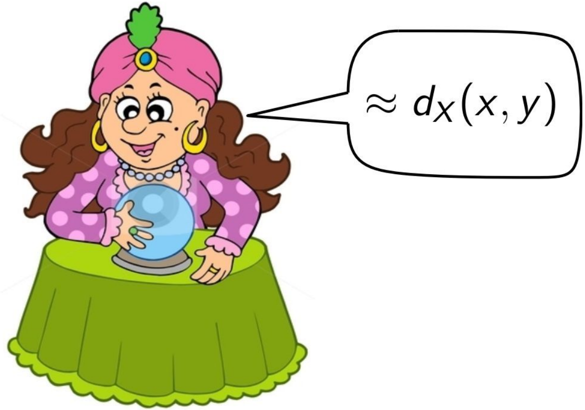

The properties of interest are size, distortion and query time.

Distance Oracles

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For every n-point metric space and $\boldsymbol{k} \geq 1$*, there is a* ***set*** $\mathcal{T}$ *of* $k \cdot n^{\frac{1}{k}}$ ***trees*** *and a* *mapping* **home** $: X \to \mathcal{T}$*, such that for every* $x, y \in X$*,*

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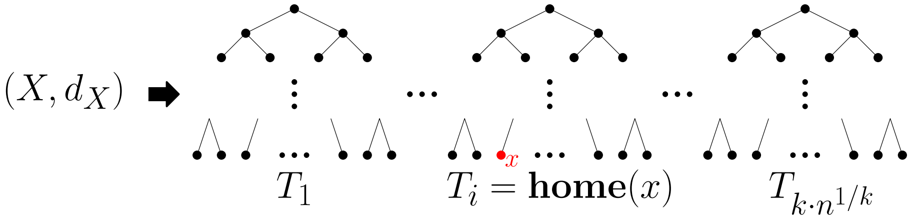

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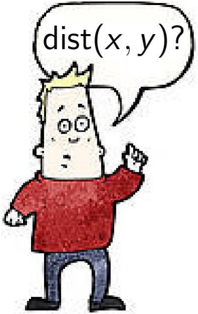

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For every ***tree metric****, there is an exact distance oracle of* ***linear size*** *and* ***constant*** *query* ***time****.*

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Theorem (Ramsey based Deterministic Distance Oracle)

For any n-point metric space, there is a distance oracle with :

Distortion	***Size***	***Query time***
$O(k)$	$O(k \cdot n^{1+1/k})$	$O(1)$

Outline of the talk - Appendix

Clan Embedding

Idea: **duplicate** vertices to meet all guarantees!

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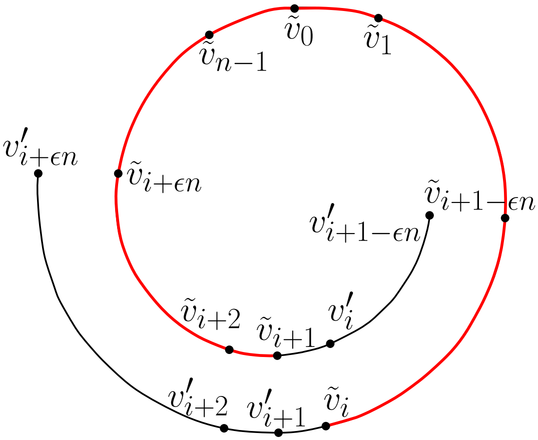

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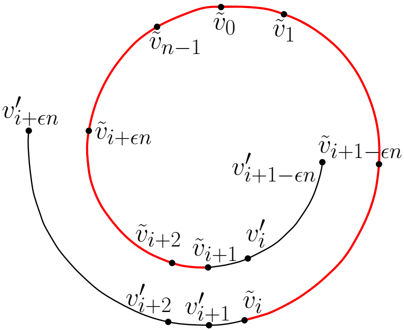

One-to-many embedding from (X, d_X) to (Y, d_Y): A map $f : X \to 2^Y$ where:

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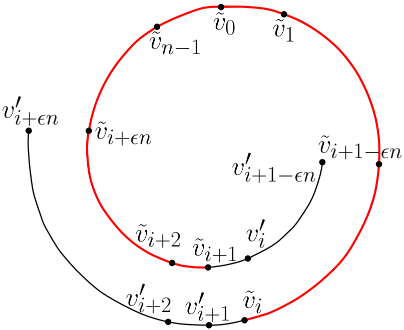

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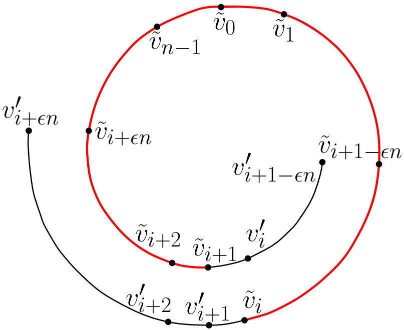

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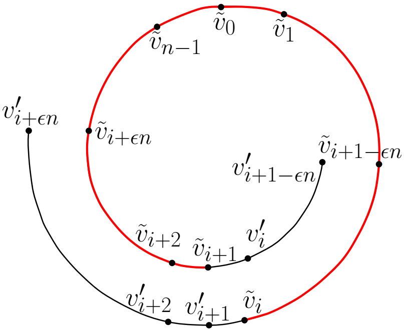

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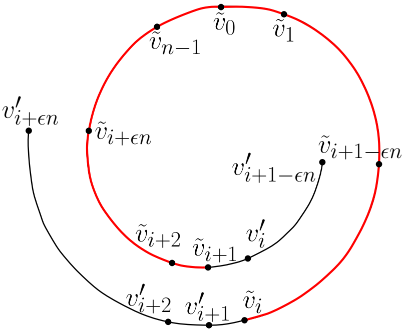

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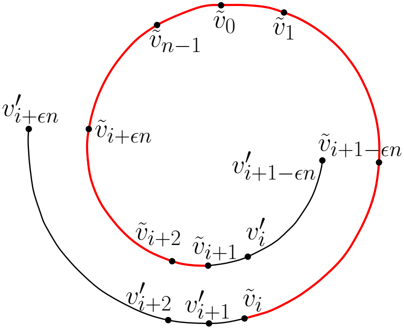

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Clan embedding is a pair (f, χ), where f is **dominating** one-to-many embedding.
(f, χ) has **distortion** t, if $\forall x, y \in X$, $\quad \min_{y' \in f(y)} d_Y(\chi(x), y') \leq t \cdot d_X(x, y)$

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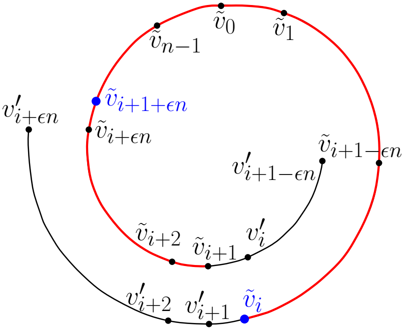
(f,χ) has **distortion** t, if $\forall x,y\in X$, $\min_{y'\in f(y)} d_Y(\chi(x),y') \le t\cdot d_X(x,y)$



$$f : C_n \to 2^{P_{2n}}, \qquad f(v_i)\subseteq\{\tilde{v}_i, v'_i\}, \qquad \chi(v_i)=\tilde{v}_i$$

$$\min_{y'\in f(v_{i+1+\epsilon n})} d_{P_{2n}}(\chi(v_i),y') \le (1-\epsilon)n$$

$$< \frac{1}{\epsilon}\cdot d_{C_n}(v_i, v_{i+\epsilon n+1})$$

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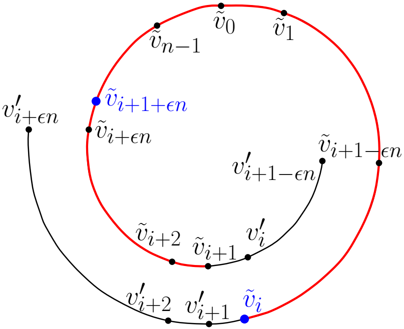

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Choose i u.a.r., then $\quad \mathbb{E}[|f(v_i)|] = 1 + 2\epsilon.$

Theorem (Clan embedding into trees, [Filtser, Le 21])

(X, d_X) *n point metric space,* $\forall \epsilon \in (0,1)$*, there is* ***distribution*** $\mathcal{D}$ *over* ***dominating clan embeddings*** *into trees such that:*

- $\forall\ (f, \chi) \in \mathrm{supp}(\mathcal{D})$ *has distortion* $O(\frac{\log n}{\epsilon})$.
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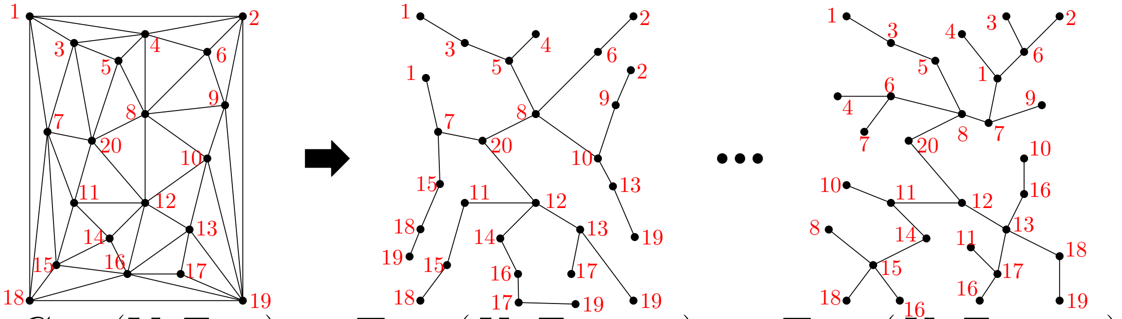

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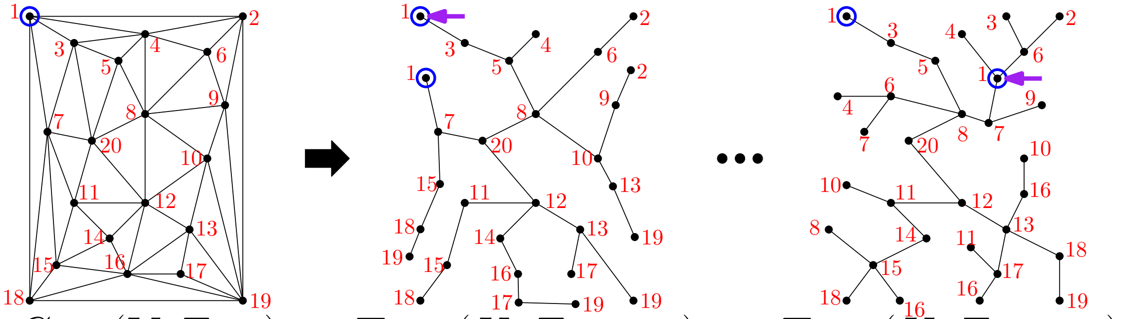

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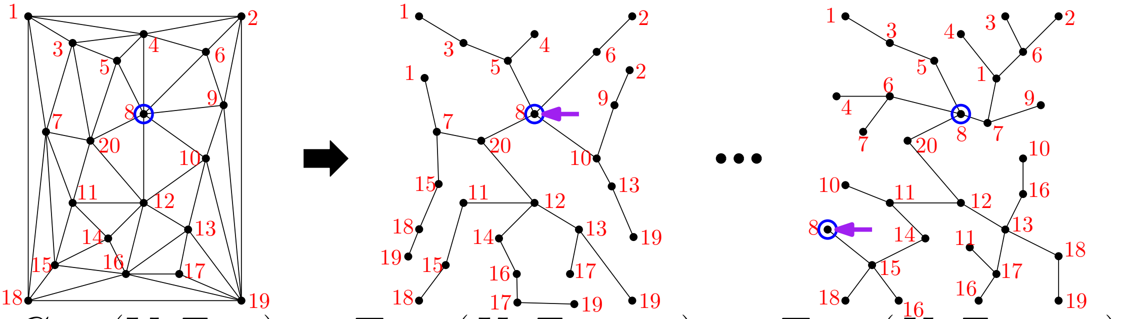

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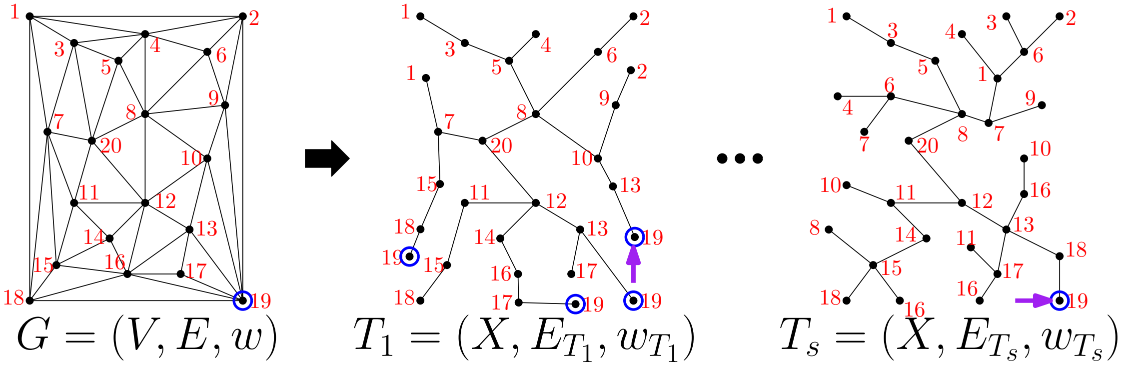

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(both) Tight!

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Compromises: Not a real classic embedding

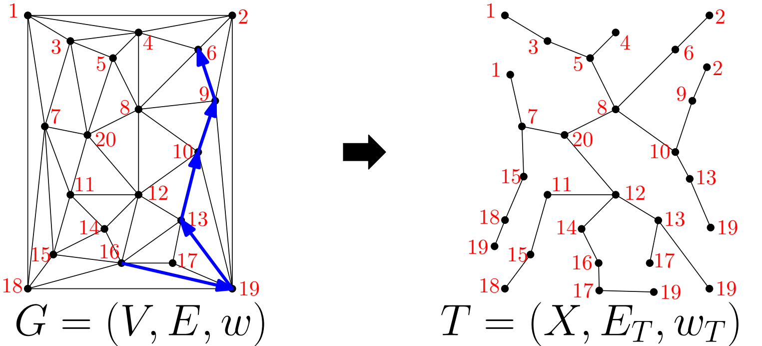
Path Distortion

Originally appeared in [Bartal, Mendel 04] in the context of multi-embeddings.

Definition

One-to-many embedding $f : X \to 2^Y$ has *path-distortion* t if for every sequence $(x_0, x_1, \ldots, x_m)$ in X there is a sequence $y_0, \ldots, y_m$ where $y_i \in f(x_i)$, and

$$\sum_{i=0}^{m-1} d_Y(y_i, y_{i+1}) \le t \cdot \sum_{i=1}^{m-1} d_X(x_i, x_{i+1}).$$



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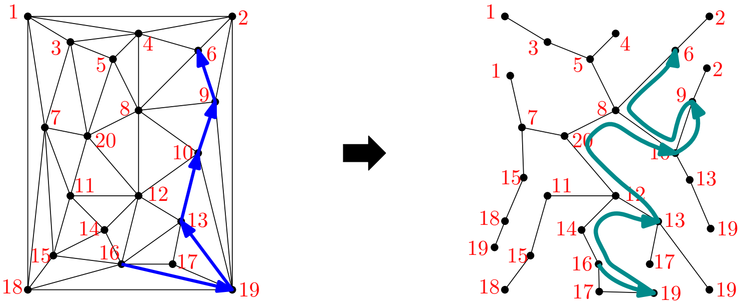

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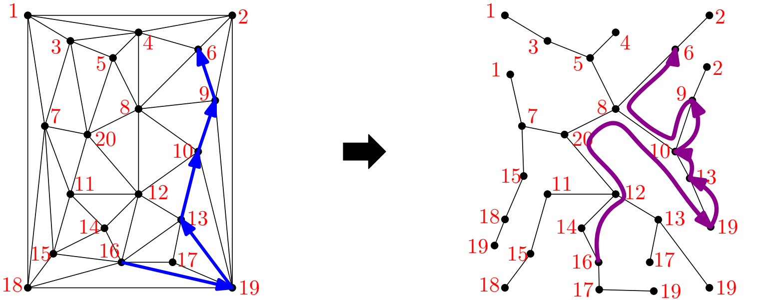

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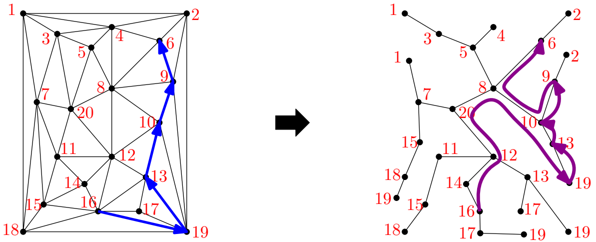

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That is there is a one-to-many embedding with a total of $O(n^{1+\frac{1}{k}})$ copies and path distortion $O(k\cdot\log n)$.

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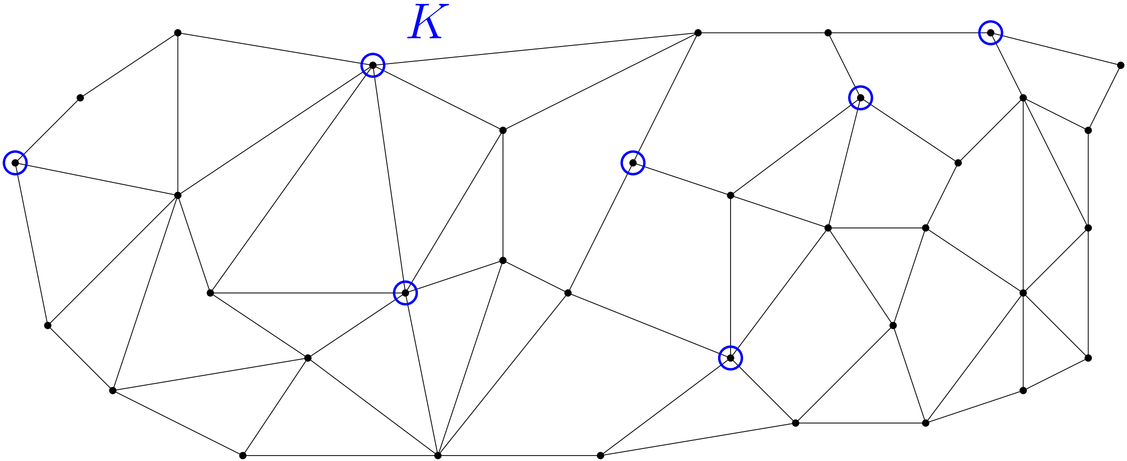
Or a total of $O(n^{1+\frac{1}{2}})$ copies and path distortion $O(\log n)$.

Outline of the talk - Appendix

Steiner Tree

Given set of terminals K, find minimum weight tree T spanning K



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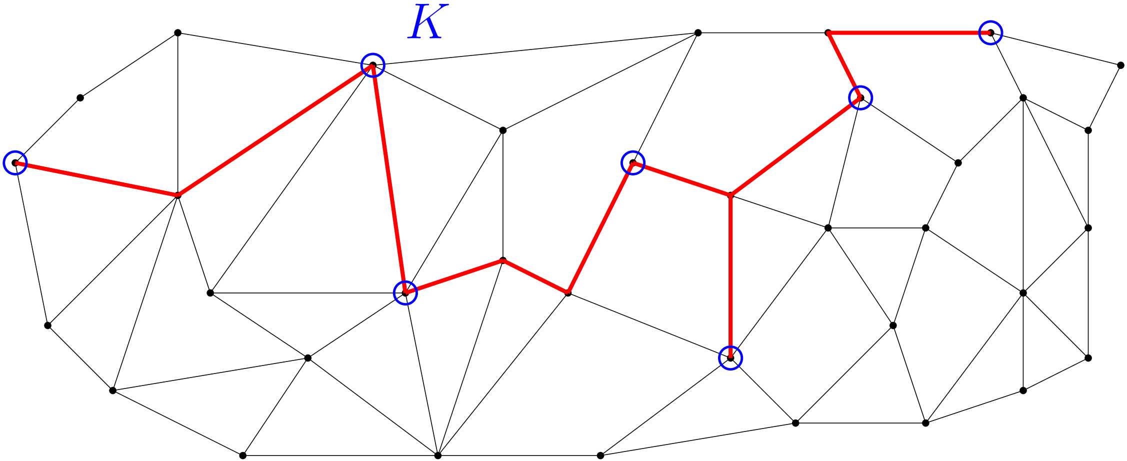

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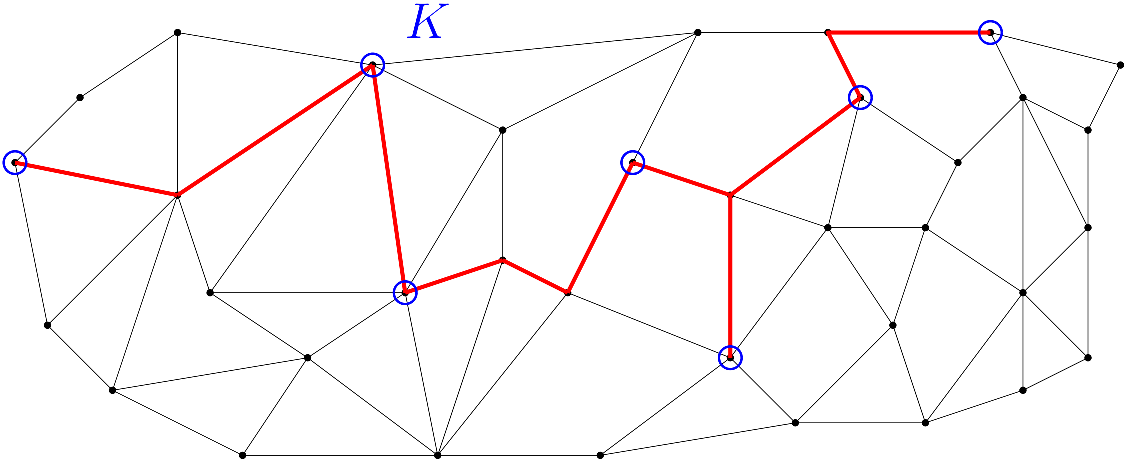
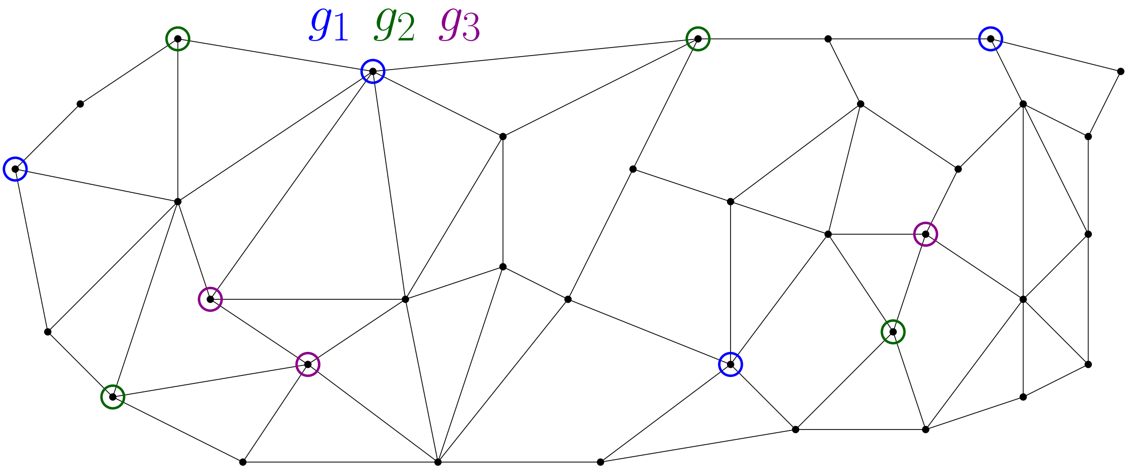


In class we saw a 2-approximation algorithm for the Steiner tree problem.

Group Steiner Tree

Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i



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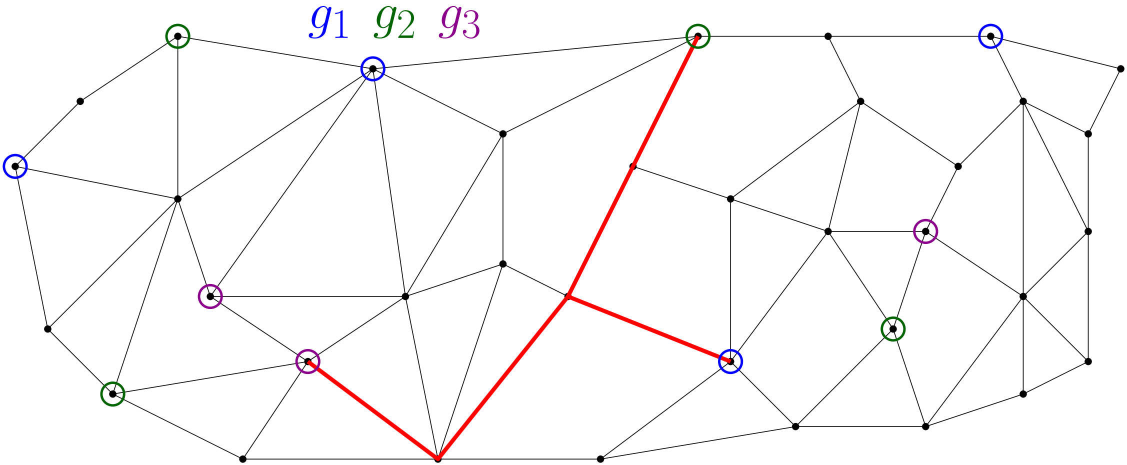

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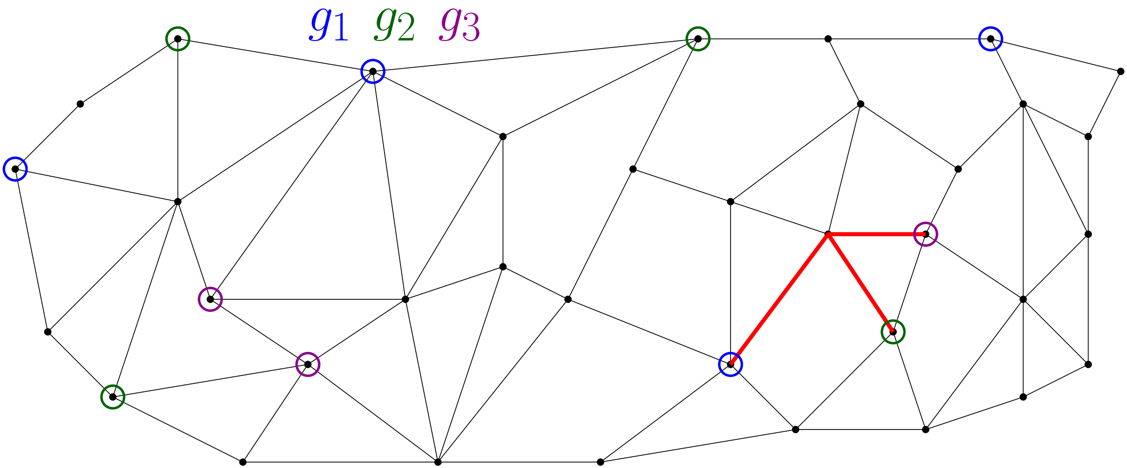

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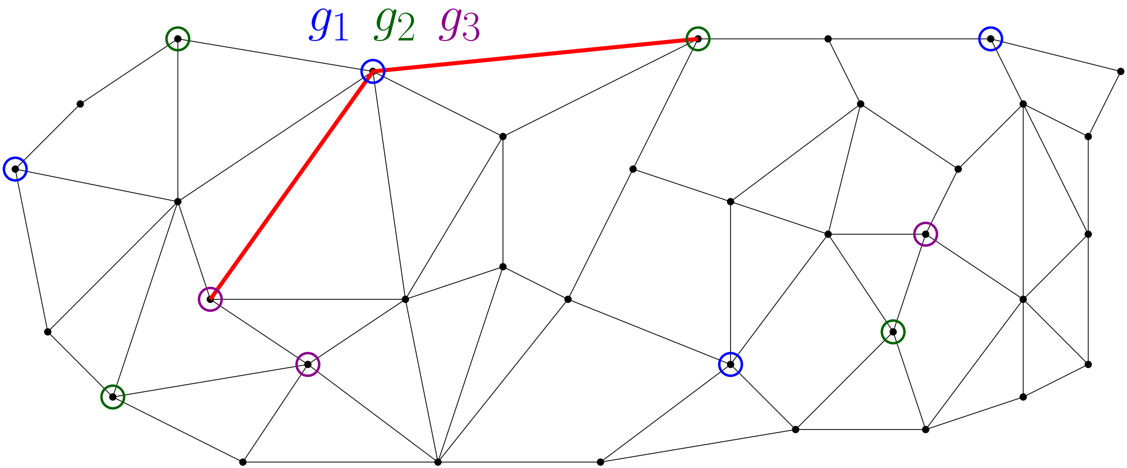

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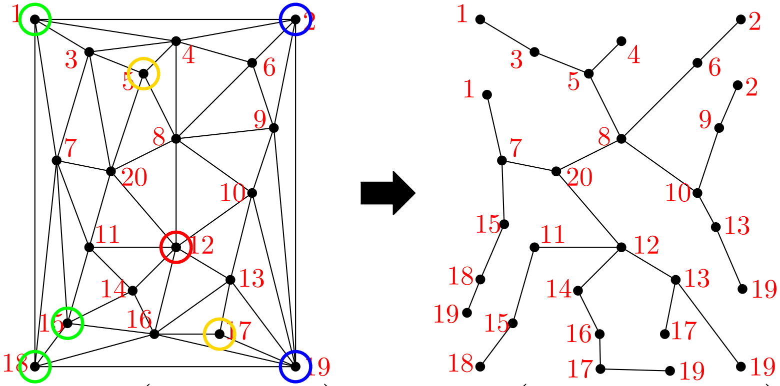

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$$g_1 \quad g_2 \quad g_3 \quad g_4$$

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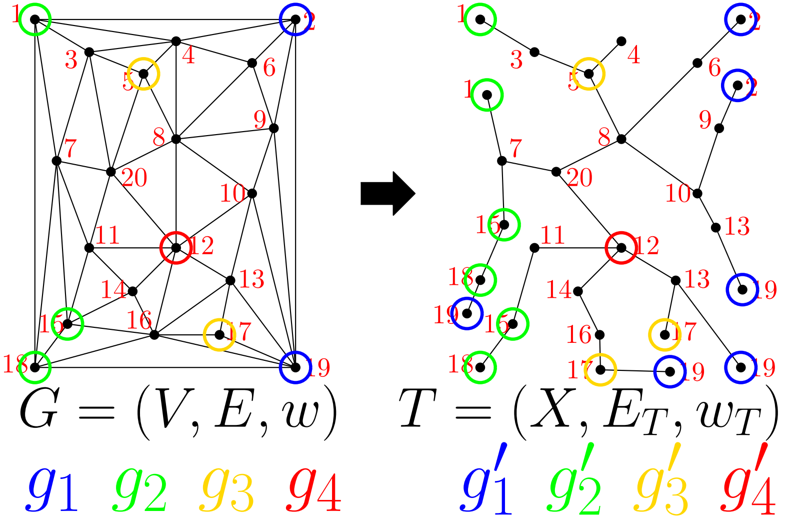

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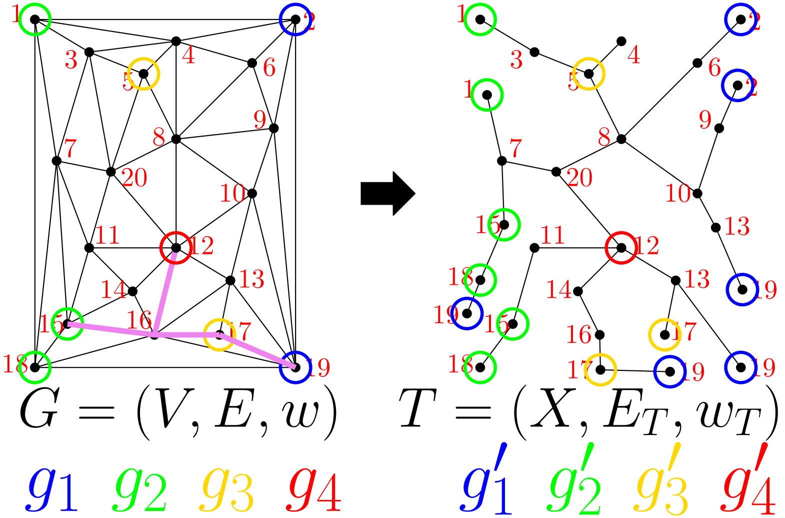

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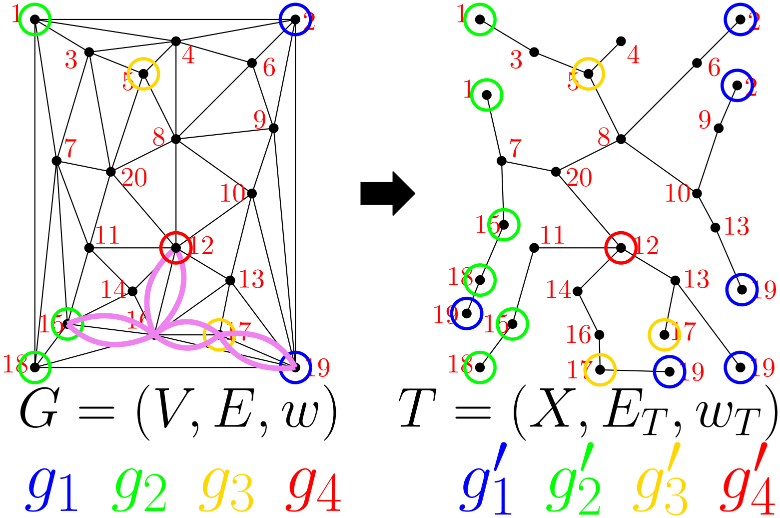

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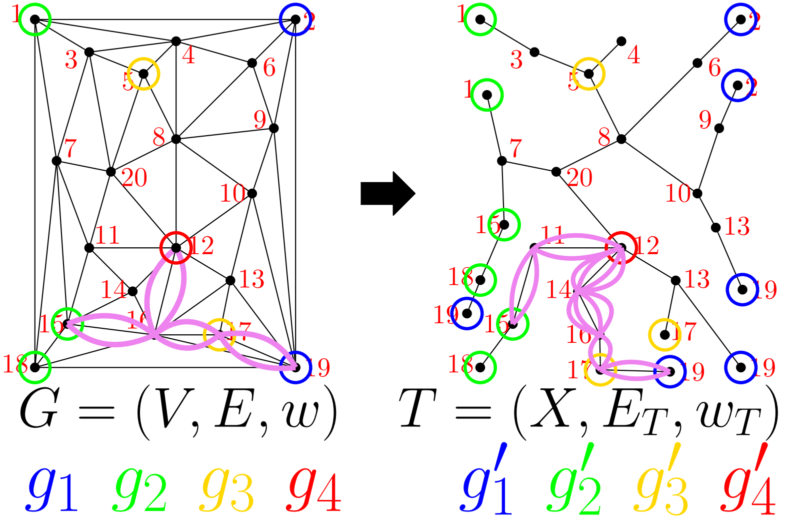
Guaranteed path: $S^\star_T$ (valid solution), $w(S^\star_T) \le O(\log n) \cdot w(S^\star)$



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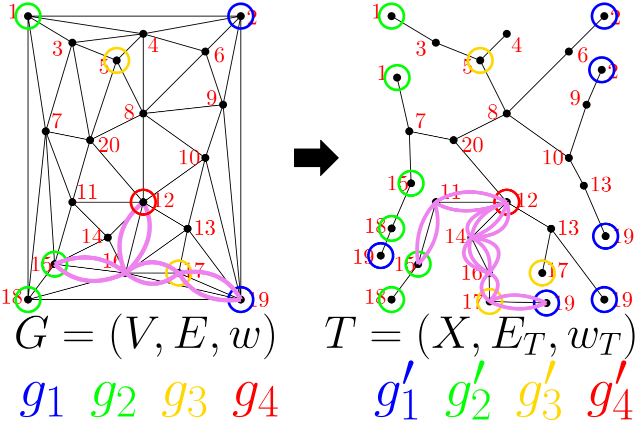

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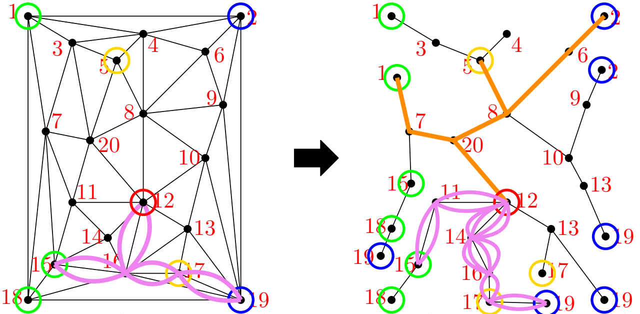

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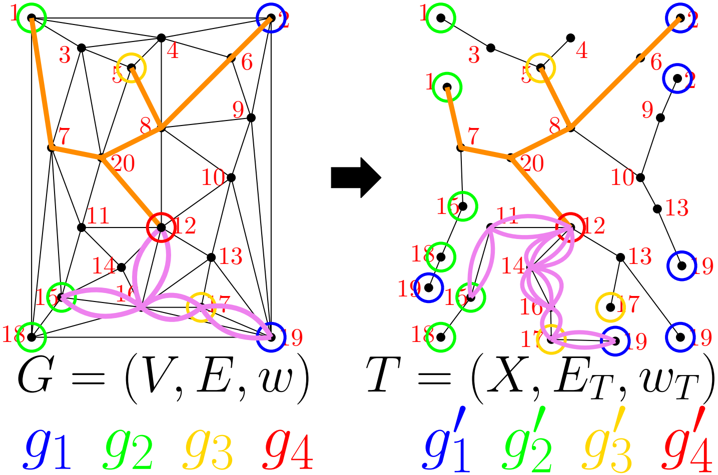
Return: $\tilde{S} = f^{-1}(\tilde{S}_T)$



$\tilde{S} = \cup_{\{v',u'\} \in \tilde{S}_T} P^T_{v',u'}$ is a union of shortest paths in G

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$S^\star$ optimal solution.

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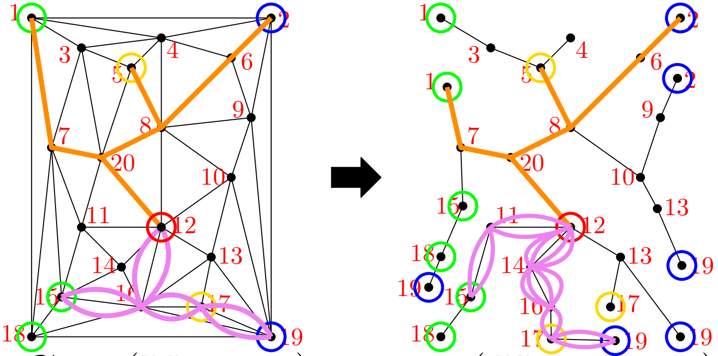

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$$w(\tilde{S}) \leq w(\tilde{S}_T) \leq O(\log^2 n \cdot \log k) \cdot w(S^\star).$$

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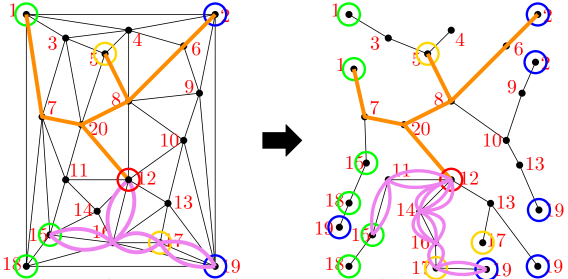

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$g_1 \quad g_2 \quad g_3 \quad g_4$ $\quad$ $g'_1 \quad g'_2 \quad g'_3 \quad g'_4$

$$w(\tilde{S}) \leq w(\tilde{S}_T) \leq O(\log^2 n \cdot \log k) \cdot w(S^\star).$$

We got an $O(\log^2 n \cdot \log k)$ **approximation**.

Clan Embeddings construction

Theorem (Clan embedding into trees, [Filtser, Le 21])

(X, d_X) n point metric space, $\forall k \in \mathbb{N}$, there is ***distribution*** *$\mathcal{D}$ over* ***dominating clan embeddings*** *into trees such that:*

- $\forall\ (f, \chi) \in \mathrm{supp}(\mathcal{D})$ *has distortion* $O(k)$.
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Definition (Ultrametric)

Ultrametric (X, d) is a metric space satisfying the **strong** triangle inequality:

$$\forall x, y, z \in X, \qquad d(x, z) \leq \max\{d(x, y), d(y, z)\} \ .$$

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(X, d_X) is a HST if X is mapped (by ϕ) to **leaves** of a rooted tree T where:

- The nodes of T associated with **monotone** labels l_v.
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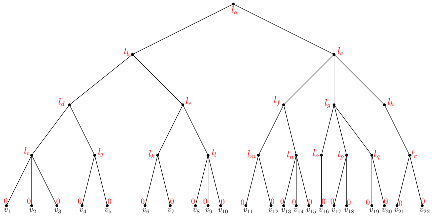

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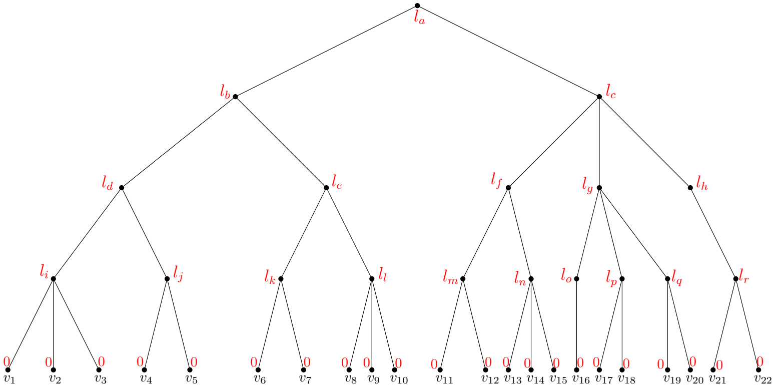

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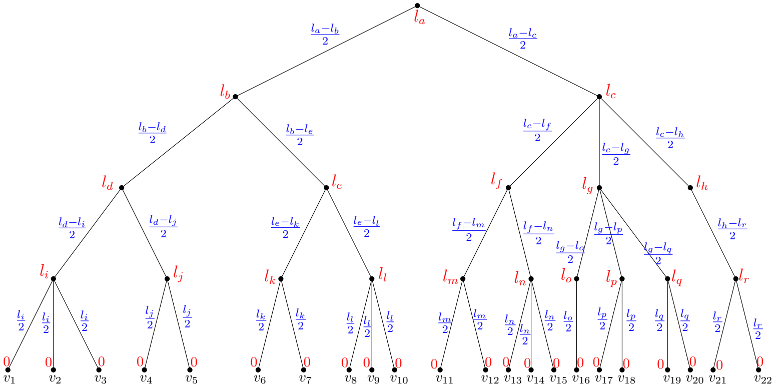
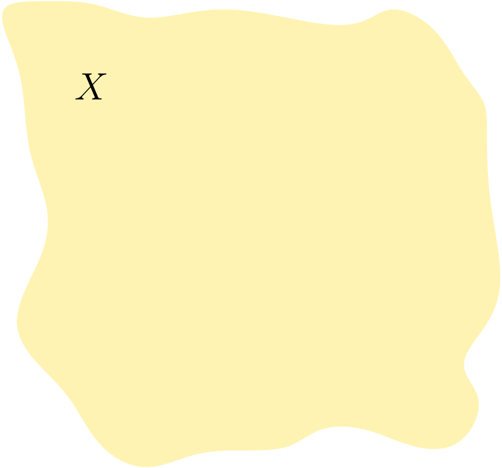
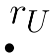


Construction



$$\ell(r_U) = \mathrm{diam}(X) = D$$



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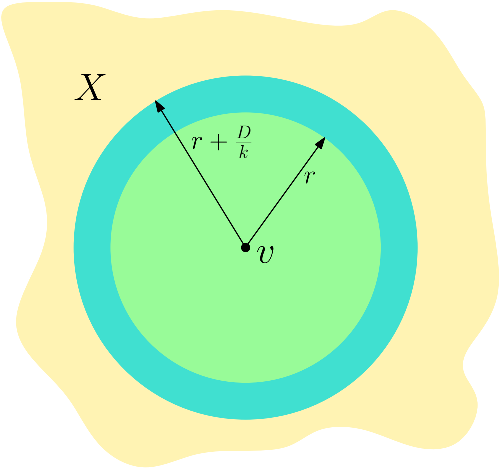

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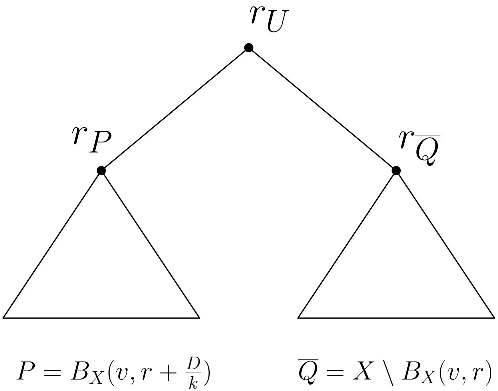

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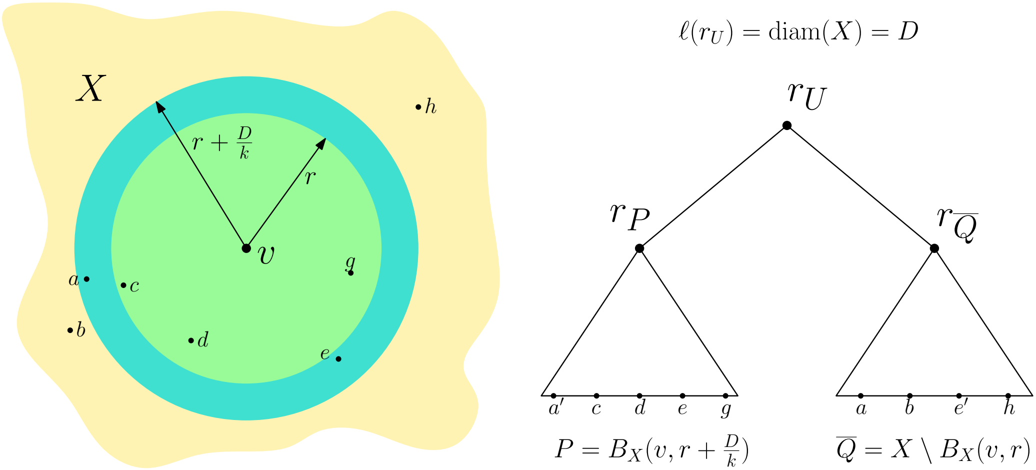

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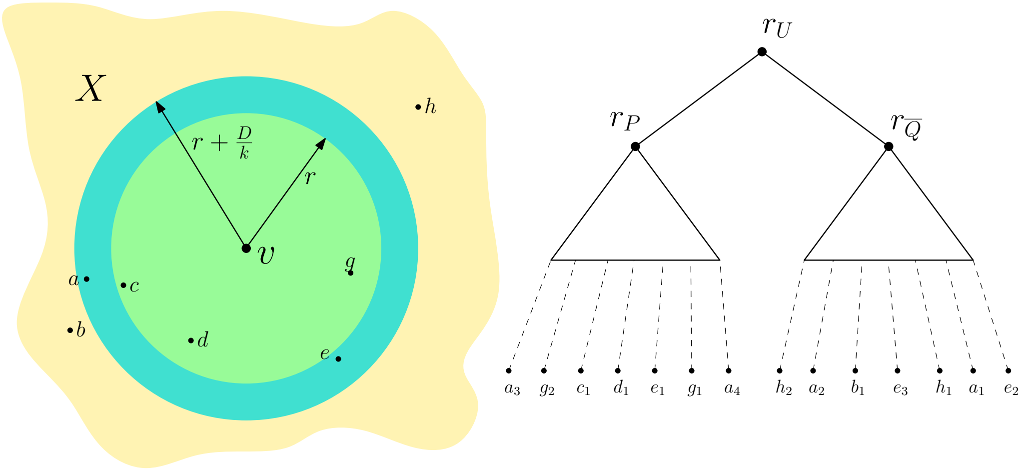

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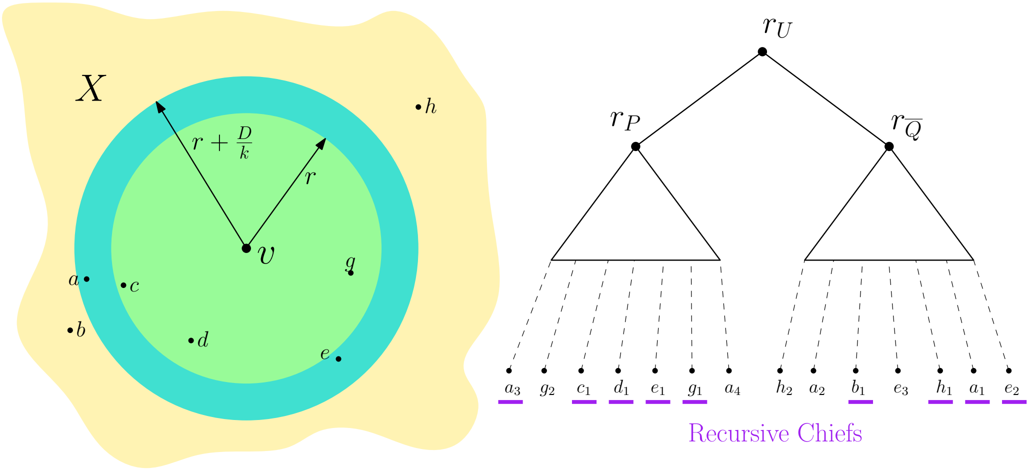

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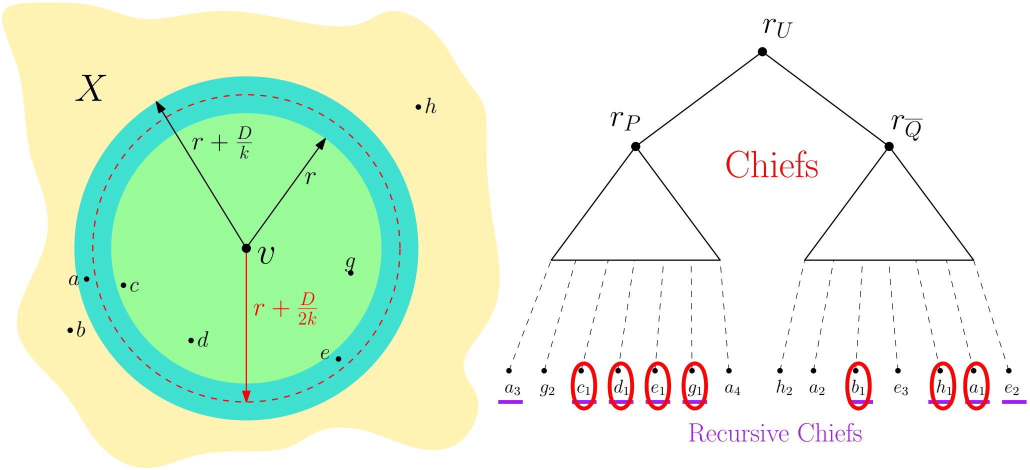
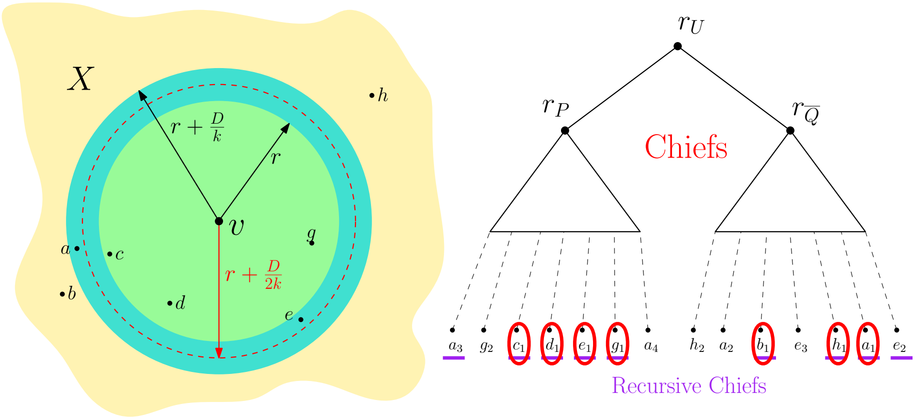
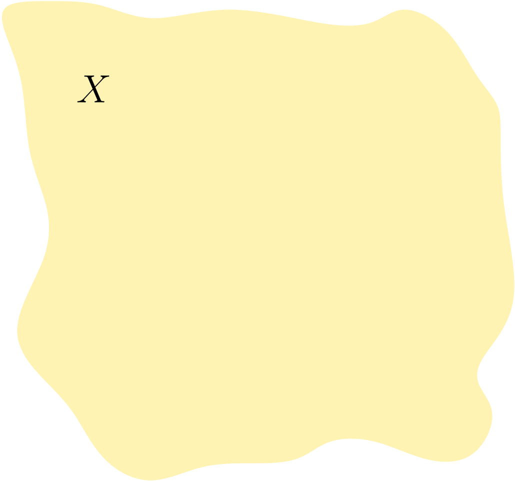


Construction - distortion bound



$$\min_{c' \in f(c)} d_U(c', \chi(a)) = D \leq 2k \cdot d_X(c, a) \ .$$

Construction - cardinality bound



$D = \mathrm{diam}(X)$

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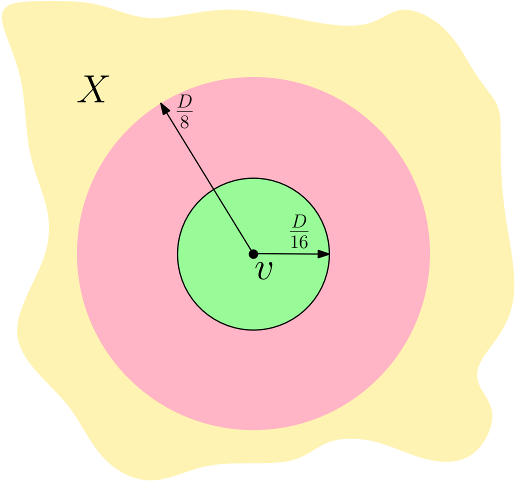

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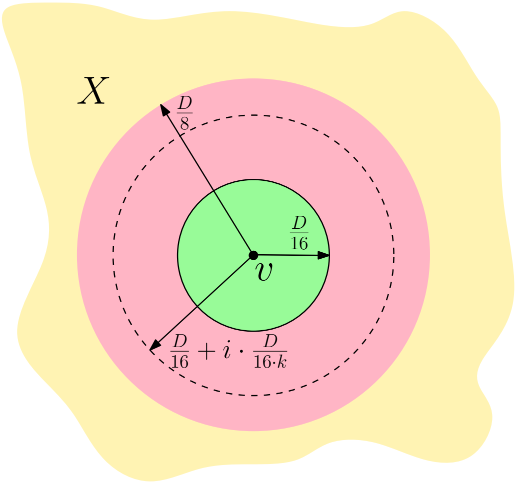

$$D = \mathrm{diam}(X)$$

$$A_i = B_X(v, \tfrac{D}{16} + i \cdot \tfrac{D}{16 \cdot k})$$
$$i \in \{0, \ldots, k-1\}$$

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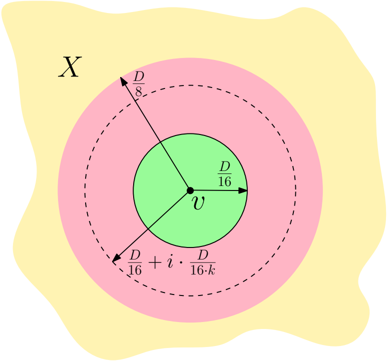

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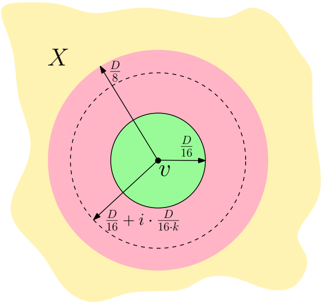

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$$|A_k| > |A_{k-1}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{1}{k}} > |A_{k-2}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{2}{k}} > \cdots > |A_0| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{k}{k}} = |A_k|$$

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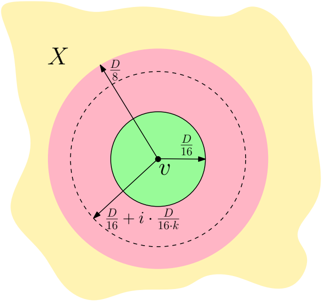

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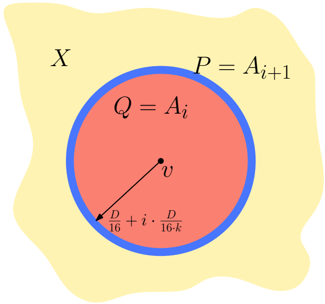

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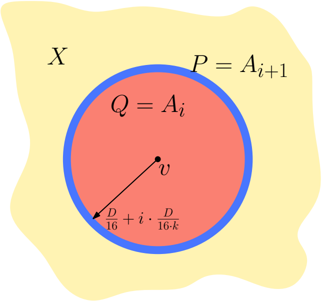

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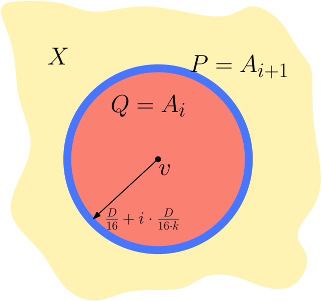

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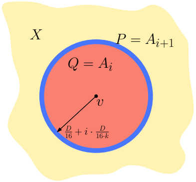

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